On the Analysis and Synthesis of Control Systems with Input Saturation

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Abstract—A saturating actuator can cause stability and performance issues such as windup for a feedback control system designed on linear methods. Anti-windup techniques can be used to preserve the linear behavior in the presence of saturation. Specifically, a local feedback around the saturating actuator via a compensator \( H \) can be used. The loop transmission \( L_n(s) \) around the actuator has been shown to be the key factor for saturation compensation. An independent \( L_n(s) \) can be created by introducing \( H \). In this paper, detailed analysis on the nonlinear responses in terms of \( L_n(s) \) and the corresponding linear responses is presented. Synthesis of \( H \) is achieved through the solution of an equivalent input disturbance rejection problem. This new synthesis scheme can accommodate the plant uncertainty and the extension to multi-input multi-output (MIMO) plants. Also, it facilitates the application of robust control tools. Examples solved using \( H_\infty \) and \( \mu \)-synthesis demonstrate the effectiveness of the proposed approach.

I. INTRODUCTION

Input saturation is inherent in all actuating devices. Actuator saturation causes performance degradation, even instability, of a control system whose design does not account for the actuator saturation nonlinearity. Saturation compensation schemes, esp. anti-windup techniques, are intended to remove or mitigate the adverse effects caused by saturation on control systems. Due to higher performance requirements of today’s control systems, actuators are pushed to operate near their physical limits and any practical control design should not ignore the possibility of actuator saturation. As a result, saturation has again gained increased attention from the controls community and a large amount of research has been generated.

In 1983, Horowitz [3] first pointed out that the system loop transmission with respect to the saturating actuator, \( L_n(s) \), determined the system behavior under saturation. Specifically, an independent \( L_n(s) \) can be created through the introduction of an additional compensator such as in the three degree-of-freedom (DOF) structure [3], see Fig. 1; then, \( L_n(s) \) can be shaped to achieve good closed-loop performance under saturation. Any saturation compensation approach, regardless of the technique and structure used, eventually modifies \( L_n(s) \) in a proper manner. So, it is \( L_n(s) \) that ultimately determines the effectiveness of saturation compensation. System behavior under saturation was described using \( L_n(s) \) in [3], where the key design equation, (10), describes how the actual plant input approaches the linear plant input after its exiting a saturation. However, that is only an approximation. In this paper, exact equations governing the nonlinear system responses in both the linear and the nonlinear modes are derived.

This idea of Horowitz was adopted in [8], using the structure in Fig. 2, to develop a robust QFT design procedure for single-input single-output (SISO) plants with uncertainty. Various design constraints for guaranteeing robust stability and performance under saturation were established for synthesizing \( H(s) \). An extension to MIMO systems is desirable. Regulating the mismatch between linear and nonlinear responses for anti-windup purposes can be found in [1], [5], and [7]. Closer examination of results in [8] reveals that design constraints are explicitly placed on closed-loop transfer functions related to plant input disturbance, cf. [6], [1]. Thus, saturation compensation may be transformed into an equivalent input disturbance rejection problem and solved by closed-loop shaping techniques. As a consequence, the plant uncertainty and MIMO plants may be treated readily.

This paper is organized as follows. Section 2 presents the governing equations for system nonlinear behaviors under saturation. Saturation compensation through input disturbance rejection design is proposed in Section 3. Examples are given in Section 4 to demonstrate the utility and efficacy of the disturbance rejection scheme by comparing to other approaches. Conclusions are given in Section 5.

II. SYSTEM BEHAVIOR UNDER SATURATION

Responses of a closed-loop system in the presence of actuator saturation alternate between the linear and nonlinear modes. In this section, we present a technique to analyze the response behaviors during the linear and nonlinear modes, for the SISO plant case. It turns out that the same dynamics dominates the response behavior at all times.

In Fig. 1 and Fig. 2, \( L_n(s) \) is the loop transmission around the actuator, \( L_n = -\frac{c}{y}, \hat{x} \) denoting the Laplace transform of \( x(t) \). Following the derivations in [3] and [8], we have the following equations

\[
\hat{\Delta}_y = \frac{\hat{\Delta}}{1 + L_n},
\]

\[
\hat{\Delta}_c = P \frac{\hat{\Delta}}{1 + L_n},
\]

where \( \hat{\Delta} = \hat{x} - \hat{\gamma}, \hat{\Delta}_y = \hat{y}_0 - \hat{\gamma}, \hat{\Delta}_c = \hat{c}_1 - \hat{c}, \hat{c}_1 \) and \( \hat{y}_0 \) are linear responses in the system in the absence of saturation.
Let \( t^i \) represent the instant of the i-th switch from linear to nonlinear mode (i.e., \( y(t) \) going into saturation) and let \( t_i \) represent the i-th switch from nonlinear to linear mode (i.e., \( y(t) \) coming out of saturation). Let \( y^a \) be the modified signal of \( y \), such that \( y^a = 0 \), \( \forall t \in [0, t^i] \); \( y^a = y \), \( \forall t \geq t^i \), and let \( y^b \) be the modified signal of \( y \), such that \( y^b = y \), \( \forall t \in [0, t^*] \); \( y^b = 0 \), \( \forall t \geq t^* \). The key equation (10) in [3] is given as

\[
\dot{y}^a - \dot{y}^b = \frac{1}{1 + L_n}[L_n(y^b - y^a)]^a,
\]

where \([L_n(y^b - y^a)]^a\) denotes the Laplace transform of \([l_n(y^b - y^a)]^a\) with \( L_n \) the Laplace transform of \( l_n \) and \( * \) denoting the convolution between two time signals. This is only an approximation for the response during the period \( t_i \leq t \leq t_i^+ \), because in general \( \Delta(t) \neq 0 \), \( \forall t \geq t_i \). It is exact only when there is no more saturation after \( t_i \).

We now start to derive exact relations effective for both the linear and nonlinear modes. According to (1), in the time domain, we have

\[
\Delta = y_i - y + l_n \ast (y_i - y).
\]

Let \( t' > t^* > 0 \). Decompose the signals at \( t^* \) for (4), we obtain

\[
\Delta^a = y_i^a - y^a + [l_n \ast (y_i^b - y^b)]^a + l_n \ast (y_i^b - y^a).
\]

Then, decompose the signals at \( t' \) for (5), we have

\[
\frac{\Delta^a}{b} = (\delta + l_n) \ast [(\dot{y}_i)^b - (y_i)^b] + [l_n \ast (y_i^b - y^b)]^b - \frac{l_n \ast [(\dot{y}_i)^b - (y_i)^b]}^b.
\]

The above equation holds for all time. Taking Laplace transform on both sides of the equation, we have

\[
(\hat{y}_i)^b - (\hat{y})^b = \frac{1}{1 + L_n}((\Delta^a)^b + \{L_n[(\hat{y}_i)^b - (\hat{y})^b]\}^a).
\]

This equation can be applied to both the linear and nonlinear modes. Now, look at the system behavior during the i-th nonlinear mode \( t_i \leq t \leq t_i, \Delta(t) \neq 0 \). Let \( t = t^i, t_i = t' \). (7) applies with \((\Delta^a)^b \neq 0 \). Next, look at the system behavior during the i-th linear mode \( t_i \leq t \leq t_i^+ \), \( \Delta(t) = 0 \). Let \( t_i = t^i, t_i^+ = t' \), we have

\[
(\hat{y}_i)^b - (\hat{y})^b = -\frac{\{L_n[\hat{y}_i^b - (\hat{y})^b]\}^b}{1 + L_n}.
\]

When \( t_i^+ \rightarrow \infty \), (8) is equal to (3).

Horowitz only focused on the plant input, however, the plant output determines the system performance, esp. when there are slow modes in the plant. Consequently we derive the corresponding equations for the plant output \( c \). From (1) and (2), we have

\[
\Delta_c = p \ast (y_i - y).
\]

Decompose the signals at \( t^* > 0 \), then at \( t' > t^* \) and taking the Laplace transform, it leads to

\[
(\Delta_c)^b = P[(\hat{y}_i)^b - (\hat{y})^b] - P[(\hat{y}_i)^b - (\hat{y})^b]^a + [P(\hat{y}_i^b - (\hat{y})^b)]^b.
\]

Combining (7), it renders

\[
(\Delta_c)^b = \frac{P}{1 + L_n}((\Delta^a)^b + \frac{P}{1 + L_n}\{P(\hat{y}_i^b - (\hat{y})^b)]^a + [P(\hat{y}_i^b - (\hat{y})^b)]^b.
\]

Assume that \( L_n(s) \) and \( P(s) \) have the same type and contain the same unstable poles if any. \( l_n(t) \) and \( p(t) \) are proportional to each other when \( t \) is large. When \( t^* \) is sufficiently large, for \( t' \gg t^* \), \( l_n(t) \) and \( p(t) \) are exchangeable with respect to the integration in the above equation, which gives the simplified approximation in the s-domain

\[
(\Delta_c)^b = \frac{P}{1 + L_n}(\Delta^a)^b
\]

The equations derived above provide insight into the nonlinear behavior of the system and form the basis for synthesis. It is evident that it is beneficial if \( L_n(s) \) can be designed independently.

**III. Saturation Compensation via Input Disturbance Rejection**

According to (1) and (2), a corresponding feedback system can be constructed from Fig. 2, see Fig. 3, of which the plant input and output response due to the deviation between the input and output of the saturating actuator are described by these two equations. Saturation compensation aims mainly to preserve the linear responses under saturation to the best possible extent, which can be achieved by minimizing the deviations between the nonlinear (with saturation) and linear (without saturation) responses of the compensated system. If \( \Delta_y(t) \) and \( \Delta_c(t) \) are kept small, the system performance under saturation is acceptable. Thus, saturation compensation may be
treated equivalently as the input disturbance rejection for the system in Fig. 3. Similar argument can be found in [6], [1]. The anti-windup compensated closed-loop system is guaranteed to be stable if the nonlinear loop in Fig. 3 is stable. The stability of the nonlinear loop can be analyzed using the Small Gain Theorem\(^1\).

Some synthesis constraints developed in [8] for SISO plants, vis-a-vis

\[
\begin{align*}
1) & \left| \frac{L_1(j\omega)}{1 + L_1(j\omega)} \right| < \gamma_1, \forall P(s), \\
2) & \left| \frac{P(j\omega)}{1 + L_1(j\omega)} + H(j\omega)P(j\omega) \right| \leq \gamma_2, \forall P(s),
\end{align*}
\]

are bounds for stability and performance of the system shown in Fig. 3, in the SISO case. They also validate the treatment of saturation compensation as a disturbance rejection problem.

If we use the compensation structure in Fig. 3 and construct \(L_n(s)\) using \(P\), \(G\), and \(H\) correspondingly, we obtain the system in Fig. 4, where \(H\) is the anti-windup compensator to be determined and \(G\) is the unconstrained controller. In Fig. 4, the system has been arranged in the general control configuration, with one exogenous input \(\Delta\) and three controlled outputs \(z_v\), \(z_{\Delta_x}\) and \(z_{\Delta_c}\). This synthesis configuration for saturation compensation requires the minimization of the overall system transfer function from \(w^*\) to \(z^*\). After including suitable weights, the control problem is to find a stabilizing \(H\) such that

\[
\|F_i(P_\alpha, H)\|_\infty \leq 1,
\]

where \(F_i\) represents a lower linear fractional transformation, [9].

IV. ILLUSTRATIVE EXAMPLES

Two simulation examples are presented to illustrate the effectiveness of the disturbance rejection approach for anti-windup compensation. The first example is taken from [8] to demonstrate the efficacy of the approach to plants with integrators and uncertainty in the parameters. The second example, based on a practical problem (cart-spring-pendulum), is taken from [4]. This example permits to show the effectiveness of the approach in a single-input two-output (SITO) plant. Notably, in the two examples, the disturbance rejection approach is easily implemented when an LFT framework is considered.

A. Example 1

The plant comprises an integrator and a second order system, the latter containing uncertainty in the parameters,

\[
P(s) = \frac{k}{s(s^2 + As + B)},
\]

where \(k \in [3.25, 6.5]\), \(A = [0.01, 0.03]\) and \(B = [2, 7]\).

A conventional two DOF QFT design was carried out for the unconstrained design, [8]. The nominal plant chosen is

\[
P_o(s) = \frac{6.5}{s^2 + 0.01s + 4},
\]

and the resultant loop transmission and filter are

\[
L_o(s) = \frac{208000(s + 5)}{(s + 2)(s + 20)(s^2 + 40s + 1600)},
\]

\[
F(s) = \frac{2}{s + 2}.
\]

The closed-loop response and unconstrained control action (bold solid) for a particular plant are also shown in Fig. 5. When the actuation signal is limited to \([-1, 1]\), the responses become oscillatory and have longer settling times (dashed).

Figure 5 also shows the responses obtained using the disturbance rejection approach (thin solid). The selected controlled outputs for design are \(\Delta_x\) and \(\Delta_c\), with the corresponding weighting functions being

\[
W_{\Delta_x}(s) = \frac{(s^2 + 6.4s + 16)}{(s^2 + 7.2s + 20.25)} \text{ and } W_{\Delta_c} = 1.
\]

Considering uncertainty on the parameters, the LFT framework for parametric uncertainty is adopted and the \(\mu\)-synthesis design is employed to determine the anti-windup controller. The resultant controller has order 7 after using model reduction techniques, see below. Wu and Jayasuriya employed the approach described in [8] for a QFT design of the saturation compensator, see below. The corresponding closed-loop time responses are shown in Fig. 5 (dash-dotted).

From Fig. 5, the \(\mu\)-synthesis design gives a time response with a faster settling time, a slower rise time, and an oscillatory

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\(^1\)The class of nonlinearities that a Small Gain analysis considers is in the sector [-1,1]. Being the deadzone nonlinearity in the sector [0,1], 'loop-shifting' can be applied to the nonlinear feedback loop, as done in [2]. Also in [1] the Small Gain analysis was relaxed to one based on the Multi-variable Circle Criterion.
transient behavior when compared to the QFT design. It is also observed that both controllers, QFT and \( \mu \)-synthesis, compel the control input to come back to linear regime faster, with the QFT controller doing it quicker.

Frequency domain plots using the \( \mu \)-synthesis anti-windup controller are shown in Figs. 6 and 7. The frequency plots illustrate the effectiveness of the disturbance rejection scheme to guarantee robust stability and performance for the plant family generated by the uncertainty in the parameters. The stability plots for the family of plants are presented in Fig. 6. In the performance frequency plots, Fig. 7, some peaks are present but they can be considered within the non-overshooting bounds.

B. Example 2 (Cart-spring-pendulum)

This example is used for comparison between the LMI-based synthesis of anti-windup controllers presented in [4] and the disturbance rejection scheme described in this paper.

The linearized model around the origin of the cart-spring-pendulum and the unconstrained LQG controller were given in [4]. The outputs of the system are the linear position of the cart \( p \) (meters) and the angular position of the pendulum \( \theta \) (radians), both taken with respect to an equilibrium position. The physical input of the system is the voltage, limited to \([-5, 5]\) Volts. The control objective in this problem is to return the pendulum and the cart quickly and gently to their equilibrium position after smaller and larger taps applied to the pendulum of length \( 2l \) at a distance \( 4/3l \) with respect to the hinge. Figure 8 shows the responses of the system for a larger pendulum tap when there is no constraint in the actuation input (bold line), significant degradation of the performance occurs when the actuator input is limited (dashed).

The synthesis of the dynamic anti-windup LMI-based compensator was done by choosing the output angular position as the controlled variable for performance. The performance and stability levels obtained for this configuration are \( \hat{\gamma}^2=181.19 \) and \( \gamma_1 = 16.62 \), respectively. The corresponding matrices for the LMI-based anti-windup controller were detailed in [4]. Figure 8 shows the improvement in the performance using the anti-windup compensation (dash-dotted).

The disturbance rejection scheme is used next. For design, the controlled outputs are \( v \), \( \Delta_x \) and \( \Delta_c \) and the corresponding weighting functions are

\[
W_v(s) = \frac{10^6(s + 1.428)}{(s + 100)} \quad I_{5 \times 5}, \quad W_{\Delta_x}(s) = 1 \quad \text{and} \quad W_{\Delta_c}(s) = I_{2 \times 2}.
\]

The resultant \( \mathcal{H}_\infty \) anti-windup controller has order 10. By using model reduction techniques the controller order is reduced to 5. The matrices of the \( \mathcal{H}_\infty \) anti-windup controller are described below. Following the analysis presented in [4] the performance level (using the same performance outputs)
$H(s) = \frac{-1.83(s - 14.85)(s + 6.13)(s^2 + 81.7s + 3049)(s^2 - 0.15s + 4.83)}{(s + 19)(s + 1.91)(s + 1.14)(s^2 + 38.9s + 1420.7)(s^2 + 1.28s + 5.98)}$ 

$H_{QFT}(s) = \frac{-200(s - 1.881)(s^2 + 3.57s + 14.5)(s^2 + 42.3s + 645.8)}{(s + 2.488)(s + 2)(s^2 + 0.5302)(s^2 + 17s + 152)(s^2 + 40s + 1600)}$

![Graph showing comparisons](image)

Fig. 8. Example 2. Comparison of the unconstrained response (bold solid) and the saturated response (dashed) to the $H_\infty$ (thin solid) and LMI-based anti-windup designs (dash-dotted).

is $\hat{\gamma} = 181.32$, almost coincident with that given by the LMI-based controller. Figure 8 shows that the performance has been improved with the use of the $H_\infty$ controller (thin line). From the plot for the position of the cart $p$ and the control input $y$, it is evident that, with the controller calculated using the disturbance rejection scheme, the system responses tend to track the unconstrained responses. Notably, the responses with the $H_\infty$ controller present less overshoot than those with the LMI-based controller. The stability level for the $H_\infty$ controller is $\gamma_1 = 2.95$.

V. CONCLUSION

In this paper, the work in [3] and [8] are revisited. We give detailed analysis of the nonlinear responses of a feedback system under actuator saturation. It complements the results originally given in [3], for the SISO case. It reveals how the system behavior under saturation is related to the loop properties and it provides insights for developing compensation schemes. Most importantly, it emphasizes the role of $L_n$ for anti-windup compensation. Based on these observations and a robust compensation methodology from [8], we propose an equivalent input disturbance rejection scheme for saturation compensation. This formulation allows the easy application of robust MIMO design techniques. Through cleverly designing the weights of the resulting minimization problem, both the saturated system performance and stability can be improved.

REFERENCES


