Chattering-free Dynamical TBG Adaptive Sliding Mode Control of Robot Arms with Dynamic Friction for Tracking in Finite-time

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Abstract — A novel continuous control system is proposed which guarantees robust tracking in finite time for uncertain robot arms subject to unknown dynamic friction. To achieve this result, several novel algorithms were combined: a - a dynamic adaptive sliding mode control; b - a time base generator to yield a time-varying gain, which allows to obtain finite time convergence; c - continuous parametrization to compensate for unknown dynamic friction, which in turn yields a continuous friction compensator; and d - a moving sliding surface, without knowledge of initial conditions, to obtain sliding mode all the time, and hence robustness. Two controllers are derived, the adaptive and the sliding mode version, both with similar closed-loop stability properties. Simulation data of a rigid robot arm is discussed.

I. INTRODUCTION

We are interested in the design of a control system, in the continuous domain, that offers a viable solution to the following problem: "Design a continuous controller for a mechanical system subject to unknown dynamic friction and unknown physical parameters, such that the error variable $e = x(t) - x_d(t) \equiv 0 \forall t \geq t_g > 0$ with measurable state $x(t)$ for any known desired trajectory $x_d(t) \in \mathbb{C}^n$, assuming that the regressor is available".

In order to solve the problem, it is required: A. smooth control input; B. compensation of dynamic friction at several velocity regimes; and C. inducing well-posed terminal attractors. To compensate for all nonlinear forces under unknown physical parameters, an accurate model of the physical system is required as well, as usually. We now discuss $A,$ $B,$ and $C$ in these subsection below, and outline the contribution in subsection $D$.

A. Chattering-free sliding modes

We elaborate further on the second order sliding mode controller proposed in [1], which stands as an easier algorithm to implement dynamic sliding mode controller in comparison to [3]-[4].

B. Compensation of dynamic friction

Dynamic friction of mechanical systems induces complex hard nonlinearities at low velocity and at velocity reversal regimes, and compensation of such forces allow high precision tracking. The discontinuous adaptive controller [6] cannot be implemented because it would require infinite bandwidth, thus a smooth or saturated version of [6] must be implemented, which renders nonzero tracking error. In this paper, we propose to implement a continuous sliding mode, similar to [1], to finally compensate for such nonzero tracking error. Note also that that [6], cannot guarantee finite-time convergence of tracking errors.

C. Finite time convergence

Dealing with tracking in finite time does not only improve the performance of the system, but it renders stronger stability properties. It is therefore not only of prime theoretical interest but also it appeals the practical implications: it guarantees that the task is exactly done as desired in a robust manner under physical (mechatronic) constraints.

The controllers based on terminal sliding mode control (TSMC) [7], [8] guarantee finite-time convergence, however practical mechatronic constraints of finite bandwidth of the actuators cannot reproduce the discontinuity, and hence [7], [8] are ill-posed. Recently, [1] proposes a well-posed dynamic TSMC to achieve finite-time convergence with terminal sliding modes that can be produced with a continuous controller. However, the convergence time is dependent on initial conditions and feedback gains, and thus is not easy to tune it up arbitrarily. In this paper, we propose to introduce a time base generator (TBG) [9] to induce well-posed terminal attractors. Since an unforsed first order differential equation based on a TBG attains a solution in finite-time [9], then the TBG sliding surface yields finite-time convergence of tracking errors. This allows to obtain an arbitrary small error at any given time.

D. Contribution

In this paper, a chattering-free dynamical terminal adaptive and a nonadaptive sliding mode controller are proposed in order to yield global stability with convergence of tracking errors in arbitrary finite time, for a rigid robot arm with dynamic friction with all physical parameters unknown. To achieve this result, a continuous parametrization of robot dynamics

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is introduced base on a second order change of coordinates such that the sliding mode condition is enforced for all time without causing chattering. To this end, we introduce a new virtual discontinuous control $\Theta_r$ such as continuous and discontinuous terms appear as follows

$$\dot{q}_r = \dot{q}_{\text{cont}} + KZ,$$

where $K = K^T \in \mathbb{R}^{x \times n}$ whose precise lower bound is to be determined yet, $sign(x)$ stands for the discontinuous signum function of vector $x \in \mathbb{R}^n$, $\kappa > 0$, and the time-varying gain $\alpha(t)$ to be defined yet using a time base generator. For large $\kappa$, $S_d$ converges monotonically to zero at time $t = t_d$ > 0, with initial condition $S_d(t_0) = S(t_0)$. Since the derivative of (9) is discontinuous, and since we want to design a continuous controller, then we add and subtract $\tanh(\lambda t, S_r)$ to $\dot{q}_r$ such that continuous and discontinuous terms appear as follows

$$\dot{q}_r = \dot{q}_{\text{cont}} + KZ,$$

where $K = K^T \in \mathbb{R}^{x \times n}$ whose precise lower bound is to be determined yet, $sign(x)$ stands for the discontinuous signum function of vector $x \in \mathbb{R}^n$, $\kappa > 0$, and the time-varying gain $\alpha(t)$ to be defined yet using a time base generator. For large $\kappa$, $S_d$ converges monotonically to zero at time $t = t_d$ > 0, with initial condition $S_d(t_0) = S(t_0)$. Since the derivative of (9) is discontinuous, and since we want to design a continuous controller, then we add and subtract $\tanh(\lambda t, S_r)$ to $\dot{q}_r$ such that continuous and discontinuous terms appear as follows

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$$\dot{q}_r = \dot{q}_{\text{cont}} + KZ,$$
where
\[
\dot{q}_{\text{cont}} = \dot{q}_d - \alpha(t)\Delta q - \alpha(t)\Delta \dot{q} + \dot{S}_d
- K\tan h(\lambda_r S_q) \tag{13}
\]
\[
Z = \tan h(\lambda_r S_q) - \text{sgn}(S_q),
\]
where \(\lambda_r = \lambda_r^T \in R_+^{n \times n}\), and (13) has the following properties: \(Z \geq -1, \ Z \leq 1, \ Z_{S_q \to 0^+} = +1, \ Z_{S_q \to 0^-} = -1\), and \(Z_{S_q \to \pm \infty} = 0\). In this way, equation \(\dot{q}_{\text{cont}}\) is continuous, and \(Z\) is discontinuous. Finally, equations (9) into (7), and (12) into (8) render
\[
S_r = S_q + K\sigma, \tag{14}
\]
\[
\dot{S}_r = \dot{S}_q + K\tan h(\lambda_r S_q) - KZ. \tag{15}
\]
Now, we design \(\alpha(t)\) of (9).

E. Time-base-generator gain \(\alpha(t)\)

The solution of an unforced TBG-based first order time-varying linear differential equation \(\dot{z} = -\alpha(t)z\) exhibits finite-time convergence, whose convergence time can be tuned arbitrarily via the time-varying gain \(\alpha(t)\) so as to drive smoothly \(z(t)\) toward its equilibrium \(z(t_0) = 0\) in finite time \(t = t_g > 0\) [9]. This property is exploited in this paper, however note that [9] is ill-posed and then a well-posed \(\alpha(t)\) is introduced here as follows
\[
\alpha(t) = \alpha_0 + \frac{\dot{\xi}}{1 - \xi + \delta} \tag{16}
\]
where \(\alpha_0 = 1 + \epsilon\), for small positive scalar \(\epsilon\) such that \(\alpha_0\) is close to 1, and \(0 < \delta \ll 1\). The generator \(\xi(\xi(t)) \in C^2\) must be provided by the user so as to \(\xi\) goes from 0 to 1 in finite time \(t_g\). The \(\dot{\xi} = \xi(t)\) is a bell shaped derivative of \(\xi\) such that \(\xi(t_0) = \xi(t_g) = 0\). Thus, the convergence time can be set arbitrarily and independently of initial conditions, in contrast to terminal sliding mode control technique wherein convergence time depends on initial conditions [7]. Notice that the solution of \(\dot{z} = -\alpha(t)z\) is \(z(t) = z(t_0)e^{\alpha(t)\delta t}\) at time \(t_g\). Thus, if we can show that \(S(t) = 0\) some time \(t\), then from (11) \(S(t) = 0 \Rightarrow \Delta q(t) = -\alpha(t)\Delta q(t)\), and accordingly tracking errors will be \(\Delta q(t) = \Delta q(t_0)e^{\alpha(t)\delta t}\) at time \(t_g\). Therefore, the control task at this stage becomes to obtain this condition (\(S(t) = 0\) for some time \(t_g\)) with a continuous controller, for the following open-loop error dynamics.

F. Open-loop error dynamics

Using equation (9)-(12), the parametrization (4) becomes
\[
Y \Theta = Y_{\text{cont}} \Theta - H(q)KZ \tag{17}
\]
with \(Y_{\text{cont}} \Theta = H(q)^T[T_{\text{cont}} + \{C(q, \dot{q}) + \sigma_{12}\} \dot{q}_r + G(q) + \frac{\sigma_0}{\alpha_0}[\dot{q}_0 + \frac{\sigma_0}{\alpha_0}]\tan h(\lambda_f S_q) + \sigma_0 \tan h(\lambda_f S_q)\], and thus the regressor \(Y_{\text{cont}} = Y_r(q, \dot{q}, \dot{q}_r, \dot{q}_{\text{cont}})\) is continuous due to \((\dot{q}_r, \dot{q}_{\text{cont}}) \in C^1\). Finally, substituting (17) into (6) yields the following open-loop error dynamics
\[
H(q)\dot{S}_r = -\{C(q, \dot{q}) + \sigma_{12}\} S_r + U - A_1 \tag{18}
\]
where \(A_1 = F + Y_{\text{cont}} \Theta + H(q)KZ\). The controller \(U\) is proposed in the following section.

III. ADAPTIVE CONTROLLER

Consider the following chattering-free dynamical adaptive terminal sliding mode controller parametrized by a time base generator:
\[
U = -K_d S_r + Y_{\text{cont}} \dot{\Theta}, \tag{19}
\]
\[
\dot{\Theta} = -\gamma Y_{\text{cont}}^T S_r, \tag{20}
\]
where \(K_d = K_d^T \in R_+^{n \times n}\), \(\Gamma = \Gamma^T \in R_+^{p \times p}\), \(\dot{\Theta}\) stands for the online estimation of \(\Theta\). Equations (19)-(20) into (18) give rise to the following closed-loop error dynamics
\[
H(q)\dot{S}_r = -\{C(q, \dot{q}) + K_f\} S_r - A_2 \tag{21}
\]
\[
\Delta \dot{\Theta} = \Gamma Y_{\text{cont}}^T S_r \tag{22}
\]
where \(A_2 = F + Y_{\text{cont}} \Delta \Theta + H(q)KZ, K_f = K_d + \sigma_{12}\), and \(\Delta \Theta = \Theta - \dot{\Theta}\). We now have the following result.

Theorem 1: The closed-loop system (21)-(22) yields arbitrarily finite-time convergence of tracking errors if \(K\) is tuned as in (32) given in the proof below. Furthermore, a sliding mode is enforced for all time, and the system attains a robust singularity-free closed-loop dynamics.

Proof. - A passivity analysis suggests that the following Lyapunov function
\[
V = -\frac{1}{2} \{S_r^T H(q)S_r + \Delta \Theta^T \Gamma^{-1} \Delta \Theta\}
\]
has the total derivative along its solution (21)-(22) as follows
\[
\dot{V} = -S_r^T K_f S_r - S_r^T H(q)KZ - \dot{V_f} \tag{23}
\]
where
\[
\dot{V_f} = \sigma_0 S_r^T [z + \sigma_0 \tan h(\lambda_f S_q)] - \sigma_0 S_r [-z h(\dot{q})]
+ \sigma_0 - \sigma_0 \tan h(\lambda_f S_q)]. \tag{24}
\]
To prove that \(\dot{V}\) is a seminonitive function, we need to show that \(\dot{V_f}\) in (24) is negative definite. To this end, consider the following derivations
\[
\dot{V_f} = \sigma_0 [S_r z - \sigma_0 \tan h(\lambda_f S_q)^T S_r] - \sigma_0 h(\dot{q}) \times S_r^T [-z + \sigma_0 \tan h(\lambda_f S_q)]
+ \sigma_0 h(q) S_r^T [-z + \alpha_0 \tan h(\lambda_f S_q)]
+ \sigma_1 \exp(\frac{\sigma_1}{\alpha_0}) \tan h(\lambda_f S_q)]
\]
\[
\dot{S}_r \leq -\alpha_0 \|S_r\|^2 + \alpha_0 h(\dot{q}) \left[ \|\dot{S}_r\| - \alpha_0 \|h(\lambda_f S_r)^T S_r\| \right]_{\geq \rho_1} + \alpha_0 h(\dot{q}) \left[ \|\dot{S}_r\| - \alpha_0 \|h(\lambda_f S_r)^T S_r\| \right]_{\geq \rho_2} + \alpha_0^{-1} \alpha_0 h(\dot{q}) \exp \left( \frac{\|S_r\|^2}{2} \|h(\lambda_f S_r)^T S_r\| \right) \geq 0
\]

where \(\rho_1 = \rho_1(S_r) > 0\), \(\rho_2 = \rho_2(S_r) \leq 1\), and (25) arises since the state \(\|\dot{S}\| \leq \alpha_0\). Then, \(\dot{V}_i\) is positive definite outside the hyperball \(\rho_0 = \rho_0(S_r) = \{S_r|Y^T r \leq 0\}\) with \(\|\rho_0\| \leq \rho\), for a positive constant \(\rho\). Thus, if we choose \(\lambda_f\) large enough, preventing that the mechatronic system does not introduce high frequency from the term \(\|h(\lambda_f S_r)^T S_r\|\), then \(\rho_0\) can be made small such that equation (25) into (23) renders

\[
\dot{V} \leq -S_r^T K_f S_r - S_r^T H(q) K Z + \rho_0. \tag{26}
\]

Note that the term \(S_r^T H(q) K Z\) is radially unbounded only when \(S_r \to \infty\), and for bounded signals it is zero only at \(S_r = 0\). Thus, according to the properties of \(Z\), and the positive definiteness of \(H(q)\) and \(K\) implies that \(\|S_r^T H(q) K Z\| \leq \eta \|S_r\|\), where \(\eta = \|H(q)||K)^1\). Then, equation (26) becomes

\[
\dot{V} \leq -S_r^T K_f S_r + \eta \|S_r\| + \rho_0. \tag{27}
\]

If \(K_f\) is large enough, and according to [7] and [13], equation (27) is semigeneric outside the boundary of \(\delta_0\), where \(\delta_0 = \{S_r|V_0 \leq 0\}\) is a hyperball centered in the origin \(S_r = 0\). For the region outside the boundaries of \(\delta_0\) we can conclude the global ultimate boundedness of error dynamics

\[
S_r \in \mathcal{L}_\infty, \Delta \theta \in \mathcal{L}_\infty \Rightarrow \|S_r\| \leq \delta_1, \tag{28}
\]

where \(\delta_1 > 0\). Then, \((S_r, \sigma) \in \mathcal{L}_\infty\), and since desired trajectories are \(\mathcal{C}^3\) and feedback gains are bounded, we have that \((\dot{q}_r, \dot{q}_{cont}) \in \mathcal{L}_\infty\), which implies that \(Y_{cont} \in \mathcal{L}_\infty\). In this way, aimed at the boundedness of the all inertial, Coriolis, gravitational matrices, and dynamic friction, equations (28) and (17) indicates that \(\|Y_{cont} \Delta \theta\| \leq \delta_2\), where \(\delta_2 > 0\) is bounded. The right hand side of (21) is therefore bounded since \(|Z| \leq 1\) inputwise, hence \(S_r \in \mathcal{L}_\infty\), which ensure that there exists a bounded scalar \(\delta_3 > 0\) such that

\[
\|\dot{S}_r\| \leq \delta_3, \tag{29}
\]

and therefore all signals of the closed-loop system are bounded. We now show that a sliding mode at \(S_r = 0\) arises for all time. To this end, consider that (15) gives rise to the following dynamical system \(\dot{S}_r = \dot{S}_r + K \text{sgn}(S_r)\), that is

\[
\dot{\tilde{S}}_r = -K \text{sgn}(S_r) + \dot{S}_r. \tag{30}
\]

The derivative of the Lyapunov function \(V_q = \frac{1}{2} S_r^T S_r\) along its solution (30) gives rise to

\[
\dot{V}_q = -S_r^T K \text{sgn}(S_r) + S_r^T \dot{S}_r \leq -\mu \|S_r\|, \tag{31}
\]

where we have used (29), and \(\mu = K - \delta_3\). Thus, we can always choose

\[
K > \delta_3 \tag{32}
\]

in such a way that \(\mu > 0\) guarantees the existence of a sliding mode at time \(t_q \leq \frac{|S_0(t)|}{\mu}\). However, notice that for any initial condition \(S_0(t_0) = 0\), and hence \(t_q \equiv 0\) implies that a sliding mode at \(S_r(t) = 0\) is enforced for all time without reaching phase, and hence (11) renders \(S(t) = S_0(t) \forall t\). If \(\kappa\) in (11) is tuned large enough such that \(S_r(t) \approx 0\) (this is achieved numerically in few samplings) for some small time \(0 < t < t_q \leq 1\), then (11) yields \(S = 0 \ \forall t \geq t_q > 0\). Thus, substituting (16) into \(S = 0\) and eliminating the independent variable \(t\) we obtain

\[
\frac{d}{dt} \Delta q = -\alpha_0 \frac{\Delta q}{(1 - \xi) + \delta},
\]

which attains the following solution

\[
\Delta q(t) = \Delta q(t_0) \left[ 1 - (1 - \xi(t))^{\alpha_0} \right] = \Delta q(t_0)^{1+\epsilon} \text{ at time } t = t_q, \tag{33}
\]

by assumption \(\xi(t_q) = 1\). Considering that \(\delta\) and \(\epsilon\) are very small, then at \(t = t_q\), tracking errors belong to a very small vicinity \(\varepsilon\) of the origin \([\Delta q, \Delta \theta]^T = [0, 0]^T\), which in practice may stand for the required precision or zero error. Note that at \(t > t_q\), the time-varying feedback gain \(\alpha(t)\) must be reset to a desired constant \(\alpha_r > 0\) at time \(t = t_q\). Now, considering that a sliding mode is enforced for all time independently of the positive value of \(\alpha(t)\), and that \(V_q > 0\) together with (31) guarantee the finite-time monotonic decreasing behavior of \(\|S(t)\|\), then for \(t > t_q\) \(\Delta q(t) \in \varepsilon \forall t > t_q\), and furthermore \(\Delta q(t)\) converges exponentially to zero since \(\Delta q(t) = -\alpha_0 \Delta q(t) \forall t > t_q\) (the exponential solution is obtained) toward \([\Delta q, \Delta \theta]^T = [0, 0]^T\). The robustness properties of the closed-loop system against bounded state-dependent disturbance \(d = d_1 + d_2 \|\dot{q}\| + d_3 \|\dot{q}\|^2 \in R^n\), where \(d_i\), for \(i = 1, \ldots, 3\), are bounded positive scalars can be addressed similarly to [13] since the invariance property is guaranteed for all time. QED

IV. SLIDING MODE CONTROLLER

Consider the following continuous control law

\[
U = -K_{\text{sat}}(S_r) - Y_{\text{cont}}(Y_{\text{cont}}^T S_r) + Y_{\text{cont}} \Theta_0, \tag{34}
\]
where \( \Theta_0 \) is an estimate of \( \Theta \), the \( p \times p \) diagonal matrix \( \hat{\Theta} \) has in its \( (i, i) \)-th entry of its diagonal \( \hat{\Theta}_{i,i} \geq |\Delta \Theta_{i,i}| \), with \( \Delta \Theta = \Theta - \Theta_0 \), \( K_d = K_d^T \in \mathbb{R}^{n \times n} \) is a positive definite matrix, and \( sat(x) = \frac{x}{\|x\|} \) is a saturation function for \( \phi > 0 \). We now have the following result.

**Theorem 2:** Consider robot dynamics (1) in closed-loop with (34). Then, closed-loop dynamics yields finite-time convergence of tracking errors if \( K_i \) is chosen as given in the proof. Furthermore, a dynamic sliding mode is enforced for all time and for any initial conditions with continuous control input.

**Proof** - Using the Lyapunov function
\[
V = \frac{1}{2} S^T \dot{S}(q) S(q),
\]
following a similar procedure to the proof of theorem 1. Details are omitted. QED

V. SIMULATIONS

The rigid model of a 2 degrees of freedom robot arm for space applications is simulated with the parameters described in table I. The desired task for the end-effector is to draw a circle of radius 0.4 m in 10 s. Initial conditions for both joint position errors are \( [\Delta q_1(t_0), \Delta q_2(t_0)] = (4.34, -4.57) \text{ deg} \).

Feedback gains are shown in table I, where nine parameters have to be tuned, being the most critical \( K \). In table I the entries of the second column stands for the scalars of the entries of the feedback gain matrices. Friction parameters represents middle hardness contact type: \( \sigma_0 = 5 \times 10^4, \sigma_1 = 2.0, \sigma_2 = 2.0, \alpha_0 = 4.0, \alpha_1 = 0.4 \) and \( \alpha_2 = 0.09 \).

In figure 1, we can observe that at time \( t_f = 1 \text{ s} \), cartesian tracking errors converge to the boundary \( \epsilon \), with \( (\Delta q_1(t_f), \Delta q_2(t_f)) = (41.2, 33.4) \text{ microns} \); and at \( t = t_f = 10 \text{ s} \) \( (\Delta q_1(t_f), \Delta q_2(t_f)) = (162, 178) \text{ nanometers} \). Note that using the adaptive controller of [14], which implements a linear sliding surface with asymptotic stability with similar control structure in static case of coordinates, yields bigger, though smoother, control effort with nonzero errors. Figure 2 shows desired trajectory, as well as the response in the cartesian phase plane \( X-Y \), for both controllers; while figure 3 depicts both controller. The highest frequency of our controller is 55.16 Hz, which can be easily obtained with commercially available robots.

When dynamic friction of the figure 4 is presented in both joints without the compensator, tracking errors does not converge anymore (figure 4, \( K \) was increased by 20% in this case). When we turned on the dynamic friction compensator, tracking errors shown in the figure 5 behave as the previous figure 1. Notice that in the lower part of figure 5, the controller depicts small smooth jumps of magnitude similar to that of the dynamic friction \( F(q, \dot{q}) \). Smooth online estimation of all inertial, mass, and composed friction parameters are obtained.

**Remarks 1:** Tuning properly the controller of theorem 2 yields similar results in comparison to theorem 1. Plots are omitted.

VI. CONCLUSIONS

A chattering-free dynamical terminal sliding mode controller which accounts for high precision tracking of a class of mechanical systems, such as robot arms with dynamic friction, is proposed. The closed-loop system gives rise to a time-varying second order sliding mode for all time, and guarantees a singularity-free terminal attractor for tracking errors, with arbitrary convergence time. This control system can be applied to a general class of second order Euler-Lagrange systems, with full actuation on each degree of freedom.

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**REFERENCES**


TABLE I
Robot parameters and feedback gains

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Fig. 1. Comparative tracking errors using our controller (19)-(20) (above) for convergence time of 1 s, and the Slotine & Li controller (below).

Fig. 2. Desired circle and their respective end-effector drawings for our controller (19)-(20), and the Slotine & Li controller without friction.

Fig. 3. Control response of our controller.

Fig. 4. When dynamic friction is presented without the compensator, tracking errors remain stable, with bigger feedback gains.

Fig. 5. Similar tracking errors are obtained when the friction compensator is turned on, however the control exhibits small jumps.