Geometry of dynamic and higher-order kinematic screws

Stefano Stramigioli∗

Herman Bruyninckx†

Dept. Inf. Tech. and Systems
Delft University of Technology
2600 GA Delft, The Netherlands

Dept. of Mechanical Engineering
Katholieke Universiteit Leuven
B-3001 Leuven (Heverlee), Belgium

http://lcewww.et.tudelft.nl/~stramigi
http://www.mech.kuleuven.ac.be/~bruyninc

Abstract

This article shows that time derivatives of twists and wrenches are indeed screws, in contrast to many classical kinematicians’ believe. Furthermore, it is proven that the “centripetal screw,” as well as the momentum of a rigid body together with all its derivatives, are also screws, and that a rigid body’s dynamics can be geometrically expressed as a screw equation.

The paper relies on a somewhat more formal treatment of screw theory than usual, in order to clarify these “controversial” issues concerning the motion of rigid systems, and in order to make the link with the more general (and historically much richer) field of differential geometry.

1 Introduction

Kinematics of rigid body motion is over two centuries old, but has never been a very unified research domain, in which two major schools can be distinguished:

• Screw theory school. Its vocabulary consists of twists and wrenches, \[1\,14\,15\]. Their foundations are the theories of Chasles \[4\] and Poinset \[10\], which prove that any velocity or finite motion of a rigid body, respectively any force exerted on a rigid body, can be decomposed in a set of two three-dimensional vectors lying on the same line (the so-called screw axis). The (scalar) ratio between the magnitudes of both parallel vectors is called the pitch of the twist or the wrench.

• Differential geometric school. It starts from the observation that rigid body motion is a special case of the more abstract concept of a Lie group \[3\,7\] , i.e., a differentiable manifold with continuous composition operator. The very rich results of the general theory of differential geometry then lead straightforwardly to corresponding properties of rigid body kinematics.

Unfortunately, few researchers in the robotics community are really familiar with the details of both schools, and this is particularly so for the screw theorists, \[8\,11\,12\]. Not in the least because the terminology and concepts used by both schools, as well as the paradigmatic background knowledge they assume, look so much different at first (and second!) sight that the inevitable correspondences between results from both research worlds remain hidden for most members of one or the other school. For example, the screw theory school is always focused on finding screw axes and pitches, and on describing their geometric and analytic properties (“axode,” “second-order screw axis,” etc.). The differential geometry school, on the other hand, seems not able to function without abundant use of “tangent bundles,” “Lie brackets,” “adjoint transformations,” “metric tensors,” or “linear connections,” which requires much more differential geometry background to understand than is strictly necessary for kinematics purposes.

The gap between both worlds closes only very slowly, mainly due to the lack of universities that teach and compare both viewpoints in the same mechanics course. This leads time and again to severe misunderstandings in discussions between members of both schools. But, more importantly, this lack of synergetic cross-fertilization slows down the evolution of the state-of-the-art in (both schools of) rigid body kinematics and dynamics.

The Ball 2000 Symposium—which took place in Cambridge (UK) in July 2000 to commemorate the famous publication \[1\] of Sir Robert Stawell Ball—was an excellent example of this scientific schism. Many of the “screw theory” papers presented there seemed not sure whether, for example, rigid body accelerations or momentum are screws or not. The following citations from a keynote paper \[6\] illustrates this uncertainty:

• “Acceleration of a rigid body is a very different thing as compared with first-order kinematic quantities, because it cannot be associated with an axis that has a pitch, there being, ordinarily, only one point of the body that, at any instant, has zero acceleration.”

• “If Ball has seen any advantage in putting momen-
tum on a screw, surely he would have done so. But he did not."

It seems that a lot of results or missing conclusions within the screw theory school could have been derived in the differential geometry school since more than a century already. For example, the Ball Symposium features such a “new” result in the form of the so-called centripetal screw \([12]\), i.e., the “screw vector product” of the velocity twist and the momentum screw; in Lie theory this is nothing else but the Lie bracket of two vector fields, one generated by the motion of the body (“velocity twist”), and one generated by mapping the momentum onto the twist space by means of the hyperbolic form \([12]\)(which is known to be the natural identification\([1]\) between twist and wrench spaces).

So, the motivations behind this paper are to prove to screw theory people that (i) higher-order time derivatives of rigid body motions have the physical properties of screws; (ii) the six-dimensional momentum vector is also a screw; and (hence!) (iii) the dynamics equation of a rigid body is a screw equation. These insights are known to some roboticists for quite some time already; our hope is to be able to present an approach that screw theorists can follow without having to learn too many new things.

2 Higher order derivatives

This paper uses a definition for a screw that is a bit more formal than the classical one, but that generalizes well for all time derivatives: a screw is a map that attaches a vector to each point \(P\) in three-dimensional space (this is a so-called vector-field), in such a way that this vector-field is uniquely defined by (i) a line and (ii) a scalar number, called the pitch. One needs to know only the (directed!) line and the pitch, in order to know the velocity \(v_p\) of any point \(P\) (with position coordinate three-vector \(p\)) on a rigid body that moves with a velocity represented by the given screw. With respect to a second right-handed orthogonal reference frame, this map’s analytical expression is:

\[
\begin{pmatrix}
    v_p \\
    0
\end{pmatrix} = \tilde{T} \begin{pmatrix} p \\ 1 \end{pmatrix}, \quad \text{with} \quad \tilde{T} = \begin{pmatrix}
    \tilde{\omega} & v \\
    0 & 0
\end{pmatrix}, \quad (1)
\]

and \(\tilde{\omega}\) a real, skew-symmetric \(3 \times 3\) matrix. Its non-zero components are the components of the angular velocity vector \(\omega\) of the body:

\(v_p = \omega \wedge p + v\).

This \(v\) is the velocity three-vector of the (virtual) point on the moving body that instantaneously coincides with the origin of the reference frame.

The above-mentioned relationships are one-to-one. Hence, any matrix of the same form as \(\tilde{T}\) represents a screw.

2.1 Time derivatives of twists

This Section analyses higher-order time derivatives like accelerations, jerk and so on.

Theorem 1 (Higher order derivatives of velocities). For a moving rigid body, the \(n\)-th order derivative of the velocity \(v_p\) (with respect to a second reference frame) of a point \(P\) belonging to the rigid body is:

\[
V^{(n)} = \tilde{T}^{(n)} P + f(\tilde{T}, \tilde{T}^{(1)}, \ldots, \tilde{T}^{(n-1)}) P, \quad (2)
\]

where \(P = (p^T, 1)^T\) are the homogeneous coordinates of the point under consideration in a frame fixed with the reference space, \(V = (v^T, 0)^T\) is the point’s velocity, \(\tilde{T}\) is the instantaneous twist of the body as expressed in Eq.(1), and \(f()\) is a function of the matrices given as its argument.

Proof by induction. The induction base is easily checked, and we will only prove the induction step. Suppose that Eq.(2) is true for \(n - 1\):

\[
V^{(n-1)} = \tilde{T}^{(n-1)} P + g(\tilde{T}, \tilde{T}^{(1)}, \ldots, \tilde{T}^{(n-2)}) P.
\]

Differentiating we get:

\[
V^{(n)} = \tilde{T}^{(n)} P + \tilde{T}^{(n-1)} \dot{P} + g\dot{P} + g\dot{\tilde{T}} P,
\]

but since \(\dot{P} = \tilde{T} P\) we obtain:

\[
V^{(n)} = \tilde{T}^{(n)} P + (\tilde{T}^{(n-1)} \dot{\tilde{T}} + h(\tilde{T}, \ldots, \tilde{T}^{(n-1)}) + g\tilde{T}) P.
\]

From the previous theorem we can draw an important conclusion: in case that \(\tilde{T} = \ldots = \tilde{T}^{(n-1)} = 0\), we have that \(V^{(n)} = \tilde{T}^{(n)} P\). Because the time derivative of a matrix of the form of \(\tilde{T}\) still has the same form, \(\tilde{T}^{(n)}\) represents a screw which maps points on the rigid body to their \((n + 1)\) derivatives.

Hence, the acceleration of any point on a moving body is the sum of (i) (what can now rightfully be called) the acceleration screw \(\tilde{\tau}^{(2)}\) of the body, and (ii) velocity-dependent terms. This is an important decoupling that classical kinematicians don’t make. They rightfully say that the acceleration of points on a moving body doesn’t have screw properties (because the velocity-dependent terms don’t have the screw property), but they mistakenly infer from this that acceleration is completely void of geometrical screw interpretation.

2.2 Higher order derivative of moments

The same kind of arguments reported for twists and their derivatives hold for wrenches too. For wrenches, we can consider:

\[
\begin{pmatrix}
    \tau \\
    0
\end{pmatrix} = \hat{W} \begin{pmatrix} p \\ 1 \end{pmatrix}, \quad \text{with} \quad \hat{W} = \begin{pmatrix}
    \tilde{j} & m \\
    0 & 0
\end{pmatrix}. \quad (3)
\]
τ is the pure momentum felt around the point p. As for twists, it is also possible to prove that
\[ \tau^{(n)} = \hat{W}^{(n)} P + f(\hat{T}, \hat{T}^{(1)}, \ldots, \hat{T}^{(n-1)}, \hat{W}, \hat{W}^{(1)}, \ldots, \hat{W}^{(n-1)}) P. \]  
(4)

This implies that, here also, \( \hat{W}^{(n)} \) has all properties of a screw. It is the screw representing the local moment variations due to a changing system of forces and momenta applied to the rigid body, in the situation in which at the time under consideration:
\[ \hat{T} = \hat{T}^{(1)} = \ldots = \hat{T}^{(n-1)} = 0, \]
\[ \dot{W} = \hat{W}^{(1)} = \ldots = \hat{W}^{(n-1)} = 0. \]  
(5)

### 2.3 Differential geometric interpretation

The previous Section has shown that, in general, a \( n \)-th order derivative of the velocity of a rigid body depends on all the derivatives of the twist up to order \( n - 1 \). It we consider \( \hat{T}(t) \) as a function of time, the space composed of all its derivatives up to order \( n \) is called the jet space of the space of \( \hat{T} \), which has, since long, been known to be isomorphic to the \( n \)-th power of the Lie algebra \( se(n) \). It is therefore not a surprise that, in a Lie group context, any derivative of a twist has again a screw structure.

The same reasoning is true for wrenches belonging also to a vector space, namely \( se^*(3) \). What changes here is that the space which should be considered for a complete information of a local moment in a point is not only function of a jet of the twists, but also of a jet of wrenches. We can still talk about jets for wrenches by considering \( se^*(3) \) as the basic manifold and defining jets on it.

### 3 The time derivative of a screw

The previous Section focused on an analytical treatment; this Section investigates the geometrical relation between a screw and its time derivatives. Consider a twist in vector form:
\[ T = \begin{pmatrix} \dot{\omega} \\ \dot{v} \end{pmatrix}. \]

We can write this twist as the product of a scalar \( m_v \) and a unit twist \( \hat{T} \):
\[ T = m_v \hat{T} \]
where
\[ \hat{T} = \begin{pmatrix} \dot{\omega} \\ \dot{v} \end{pmatrix}, ||\dot{\omega}|| = 1, \text{ or } \hat{T} = \begin{pmatrix} 0 \\ \dot{v} \end{pmatrix}, ||\dot{v}|| = 1. \]

Taking the time derivative we obtain:
\[ \dot{T} = \dot{m}_v \hat{T} + m_v \hat{T}, \]
where the only element which is left to be analyzed is \( \hat{T} \).

We have two cases, \( \hat{T} \) is of the form \((0 \ a^T)^T \) or it is of the form
\[ \hat{T} = \begin{pmatrix} a_0' \\ a_v' \end{pmatrix}. \]

In the first case we are done and we just obtain a line at infinity; the second case gives a finite line and we should calculate its place and orientation. In this case, we have the following well-known screw decomposition for \( \hat{T} \):
\[ \hat{T} = \begin{pmatrix} \omega \ r \wedge \omega \\ h \end{pmatrix} + \begin{pmatrix} 0 \\ \omega \end{pmatrix}. \]

We now decompose \( \hat{T} \) as a product of its module and a unit element as defined previously:
\[ \hat{T} = m_a \begin{pmatrix} a_0 \\ a_v \end{pmatrix}, \]
where \( ||a_0|| = 1 \). We should find a screw representation for the acceleration and therefore we should find \( \ddot{r}, \ddot{\omega}, \ddot{h} \) in the following expression:
\[ \begin{pmatrix} a_0 \\ a_v \end{pmatrix} = \begin{pmatrix} \ddot{\omega} \\ \ddot{r} \wedge \ddot{\omega} \end{pmatrix} + \ddot{h} \begin{pmatrix} 0 \\ \omega \end{pmatrix}. \]

We can directly conclude that:
\[ \ddot{\omega} = a_0, \]
which implies that the axis of the screw \( \hat{T} \) is orthogonal to the axis of \( T \) since \( a_0 \) is the derivative of a unit vector in the direction of \( \omega \). Furthermore, we still need:
\[ a_v = \ddot{r} \wedge \ddot{a}_0 + \ddot{h} a_0, \]
(6)

where 1 is orthogonal to \( a_0 \) and 2 is along \( a_0 \). Furthermore we have that
\[ m_a a_v = \ddot{r} \wedge \ddot{\omega} + \ddot{\omega} \wedge \ddot{\omega} + \ddot{h} \ddot{\omega} + \ddot{h} \ddot{\omega}. \]
(7)

To equate Eq. (6) and Eq. (7), we can see that, since \( a_0 \) is orthogonal to \( \ddot{\omega} \) and \( a_0 \) is parallel to \( \ddot{\omega} \), \( b, d \) correspond to the term 1, \( c \) to the term 2 and \( a \) in general to both. We can therefore split \( a \) in a part belonging to 1 and a part belonging to 2:
\[ \ddot{r} \wedge \ddot{\omega} = \frac{\ddot{\omega} \wedge \ddot{\omega}}{||\ddot{\omega}||^2} + \frac{\ddot{\omega} \wedge ((\ddot{r} \wedge \ddot{\omega}) \wedge \ddot{\omega})}{||\ddot{\omega}||^2}. \]

The first of the previous terms belong to 2 and the second to 1. We obtain therefore that:
\[ \ddot{r} \wedge \ddot{\omega} = \frac{\ddot{\omega} \wedge ((\ddot{r} \wedge \ddot{\omega}) \wedge \ddot{\omega})}{||\ddot{\omega}||^2} + r \wedge \ddot{\omega} + \ddot{h} \ddot{\omega}, \]
and
\[ \ddot{\mathbf{r}} = \dot{\mathbf{r}} + \frac{(\dot{\mathbf{r}} \wedge \dot{\omega})^T \dot{\mathbf{r}}}{||\dot{\omega}||^2}. \]

This implies that:
\[ \ddot{h} = h + \frac{(\dot{\mathbf{r}} \wedge \dot{\omega})^T \dot{\mathbf{r}}}{||\dot{\omega}||^2}, \]

and
\[ \ddot{\mathbf{r}} = \dot{\mathbf{r}} + \frac{\dot{\omega} \wedge (\dot{\mathbf{r}} \wedge \dot{\omega})}{||\dot{\omega}||^2} + \frac{\dot{\omega} \wedge \dot{\omega}}{||\dot{\omega}||^2}, \]

or
\[ \ddot{\mathbf{r}} = \dot{\mathbf{r}} + \frac{\dot{\omega}}{||\dot{\omega}||^2} \wedge (\dot{\mathbf{r}} \wedge \dot{\omega} + \ddot{\mathbf{r}}). \]

All this still holds from the \( n \)-th derivative to the \((n+1)\)-st one.

## 4 Rigid body momentum

The momentum \( \mathcal{P} \) of a moving body, with (positive-definite) inertia matrix \( \mathcal{I} \) and moving with an instantaneous velocity twist \( T \), is the product of \( \mathcal{I} \) and \( T \). In the differential geometry literature, it has been known since long that (i) such a positive-definite matrix induces a one-to-one mapping between its domain space (twist space \( \text{se}(3) \)) and image space (momentum space), and (ii) the domain and image spaces are dual (i.e., momentum is a linear map from twists to the real line). Hence, for a geometer, it is obvious that the momentum of a rigid body has the physical properties of a screw, and, more in particular, those of a wrench (=duals of twists)

Kinematicians want (rightfully so) to know where the momentum’s screw axis lies, what its pitch is, and how these are related to those of the body’s velocity twist. The following paragraphs explain how to find this out.

The inertia matrix has its simplest form in the so-called centroidal body-fixed reference system:
\[ \mathcal{I} = \begin{pmatrix} J & 0 \\ 0 & mI \end{pmatrix}, \]

where \( J \) is a diagonal matrix. (In any other frame, the zeros disappear, but the top-left entry of the inertia matrix remains symmetric.) The corresponding momentum then looks like:
\[ \mathcal{P} = \mathcal{I} T = \begin{pmatrix} J \omega \\ m(r \wedge \omega) + \alpha m \dot{\omega} \end{pmatrix}, \]

because, according to Chasles’ theorem [4], the twist \( T \) of the body has the following screw decomposition:
\[ T = ||\omega|| \begin{pmatrix} \dot{\omega} \\ r \wedge \dot{\omega} \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} \dot{\omega} \\ r \wedge \dot{\omega} + \alpha \dot{\omega} \end{pmatrix}. \]

Since differential geometry learns us that \( \mathcal{P} \) is dual to twists, it has the following screw decomposition (in accordance with Poinset \[10\]):
\[ \mathcal{P} = ||p|| \begin{pmatrix} \dot{\mathbf{r}} \wedge \dot{\mathbf{r}} \\ \dot{p} \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \] 8

This also implies that
\[ ||p|| = ||m(r \wedge \omega) + \alpha m \dot{\omega}||. \]

Without loss of generality, \( r \) can be taken orthogonal to \( \omega \); and \( \dot{k} = \dot{\mathbf{r}} \wedge \dot{\omega} \) can be found such that \((\dot{k}, \dot{\omega}, \dot{\mathbf{r}})\) is a right-handed orthonormal triple. Therefore:
\[ ||p|| = m(||r|| ||\omega|| \dot{\mathbf{r}} + \alpha \dot{\omega}||), \]

and
\[ ||p|| = m \sqrt{||r||^2 ||\omega||^2 + \alpha^2}. \]

In case the motion was a pure rotation:
\[ ||p|| = m ||r|| ||\omega||. \]

This is consistent with the usual notion of mechanics: for a pure rotation of magnitude \( ||\omega|| \) of a rigid body with mass \( m \) around an axis at a distance \( ||r|| \) from its center of gravity, its instantaneous translational momentum is \( m ||r|| ||\omega|| \). A pure translation, however, gives:
\[ ||p|| = ma. \]

This again confirms the intuition, since for a pure translation \( \alpha \) corresponds to the norm of the translational velocity of the rigid body. The direction of the momentum line, given by \( p \), is found from:
\[ p = m(||r|| ||\omega|| \dot{\mathbf{r}} + \alpha \dot{\omega}), \]

and therefore the line lies on the plane spanned by \( \dot{\mathbf{r}} \) and \( \omega \) and passing through \( \ddot{\mathbf{r}} \). The exact direction depends on the ratio:
\[ \frac{||r|| ||\omega||}{\alpha}. \]

It is now possible to state the following theorem which gives a geometrical interpretation of the momentum screw.

**Theorem 2.** Given a twist \( T^i \) expressing the motion of a body with inertia tensor \( \mathcal{I}^i \) with respect to the inertial frame and numerically expressed in the principal inertial frame of the body, the momentum screw is represented by a line at \( \ddot{\mathbf{r}} \) in the principal inertial frame directed toward \( \dot{\mathbf{r}} \) with pitch \( \beta \) where:
\[ \ddot{\mathbf{r}} = \frac{p \wedge J \omega}{||p||^2} = \frac{p \wedge J \omega}{m^2(||r||^2 ||\omega||^2 + \alpha^2)}, \] 9
where $p^0,i$ is the momentum of body $i$, and $W^{0,i}$ is the total wrench applied to body $i$. (The previous equation also directly shows why we chose the decomposition of Eq. (8) for the momentum: it is physically the same thing as a wrench.) Using basic matrix calculus and Lie groups concepts we see that:

$$\dot{p}^0,i = W^{0,i} \Rightarrow\begin{pmatrix} Ad_{T^0_i}^T & P^i \\
\end{pmatrix} = \begin{pmatrix} Ad_{T^0_i}^T & P^i + Ad_{H^0_i}^T \dot{P}^i = W^{0,i} \\
\end{pmatrix},$$

where $H^0_i \in SE(3)$ is the homogeneous point coordinate transformation matrix from fixed space coordinates to body coordinates, and the “adjoint representation of $SE(3)$” $Ad_{H^0_i}$ is the transformation for velocity screws generated from $H^0_i$. The derivative of $Ad_{H^0_i}$ is known to be:

$$ad_{T^0_i}^T Ad_{H^0_i}^T P^i + Ad_{H^0_i}^T \dot{P}^i = W^{0,i},$$

with $T^{0,i}$ the twist of the fixed space with respect to the body and expressed in the coordinates of the fixed space, and $ad_T$ is the “adjoint representation of $se(3)$” mapping from twists to twists. If we now multiply on the left by $Ad_{T^0_i}$ we get:

$$\begin{pmatrix} Ad_{T^0_i}^T, i & P^i & P^i \\
\end{pmatrix} = \begin{pmatrix} Ad_{T^0_i}^T, i & P^i & P^i \\
W^i \\
\end{pmatrix},$$

where $T^{i,i}_0$ is the twist of the fixed space with respect to the body and expressed in body coordinates. And therefore:

$$\dot{P}^i = -ad_{T^{0,i}}^T P^i + W^i.$$  

Because $-T^{0,i}_0 = T^{i,0}_i$, where $T^{i,0}_i$ is the twist of the body with respect to the fixed space expressed in body coordinates, we have that:

$$-ad_{T^{0,i}}^T = ad_{-T^{0,i}}^T = ad_{T^{0,i}},$$

and

$$\dot{P}^i = ad_{T^{0,i}}^T P^i + W^i.$$  

Furthermore, it can be shown that:

$$ad_{T^{0,i}}^T P^i = \begin{pmatrix} -\dot{\omega}^k_i & -\dot{v}^k_i & P^i \\
0 & -\dot{\omega}^k_i & P^i \\
\end{pmatrix} = \ldots \begin{pmatrix} P^i \omega \\
F^i \\
\end{pmatrix},$$

where

$$\mathcal{P}^i := \begin{pmatrix} P^i \omega \\
F^i \\
0 \\
\end{pmatrix},$$

and

$$\mathcal{P}^0 = F^0.$$  

This can be generalized to rigid bodies:

$$\dot{p}^0,i = W^{0,i},$$

5 Screw equations of a rigid body

It is now possible to formally prove that the dynamics equation of a rigid body is describable using screw theory if they are expressed using the momentum screw introduced in Sec.4.

Theorem 3 (Screw body dynamics). The dynamics of a rigid body moving in space can be expressed using screws as:

$$s_p = s_p \wedge s_t + s_w,$$

where $s_w$ is the screw representing the applied wrench to the body, $s_t$ is the screw representing the twist of the body, $s_p$ is the momentum screw introduced in Sec.4, the operator $\wedge$ is the so-called motor product \[2, 14\] and $s_\dot{p}$ is a new screw equal to the time derivative of the screw $s_p$ as observed from an observer fixed to the moving body.

Proof. For the proof, we will use tools of Lie group theory which turns out to be more handy for calculations. (These are, unfortunately, very little, or rather, never, used by screw theorists). Newton’s law for a point mass says that

$$p^0 = F^0.$$  

Furthermore, the angle $\theta_p$, between the axes of the twist and the momentum screws is equal to

$$\theta_p = \arctan 2(\alpha, ||r||||\omega||),$$

and the pitch of the momentum screw is equal to

$$\beta = ||J\omega|| \cos \theta_h,$$

where $\theta_h$ is the angle between the angular momentum and the translational momentum.

Proof. By substitution we can see that:

$$\ddot{r} \wedge p + \beta \ddot{p} = \frac{1}{||p||^2} (p \wedge J\omega \wedge p + (J\omega)^T \ddot{p} \ddot{p} =$$

$$\frac{1}{||p||^2} (p \wedge J\omega \wedge p + (J\omega)^T p p + (J\omega)^T \ddot{p} \ddot{p} = J\omega.$$  

\[12\]  

\[p = m(||r||||\omega||\dot{k} + \alpha \omega), \]

\[\beta = (J\omega)^T \ddot{p}. \]

Furthermore, the angle $\theta_p$, between the axes of the twist and the momentum screws is equal to

$$\theta_p = \arctan 2(\alpha, ||r||||\omega||),$$

and the pitch of the momentum screw is equal to

$$\beta = ||J\omega|| \cos \theta_h,$$

where $\theta_h$ is the angle between the angular momentum and the translational momentum.
is a skew-symmetric tensor called a Lie-Poisson tensor. To conclude, we can write the equation of a rigid body as:

$$\dot{p}^i = (P^i \wedge)T^{i,0}_i + W^i.$$  \hspace{1cm} (15)

In this matrix product, one easily recognizes Eq. (13), changing matrix representations of screws into six-vector representations of screws, and by defining the six-vector wedge product in that equation by means of the matrix product $(P^i \wedge)T^{i,0}_i$ in the equation above. This can be formally calculated using the results presented in [12]; we can calculate the screw vector product of Eq. (13) analytically by translating it to an operation on the Lie algebra $se(3)$:

$$s_p \wedge s_t \leftrightarrow \delta[\delta^{-1}P^i; T^{i,0}_i]$$

where $\delta$ is the hyperbolic form which can be used to associate twists and wrenches represented by the same geometrical screw, and $[,]$ is the commutator on $se(3)$. With trivial calculation we can than see that:

$$\delta[\delta^{-1}P^i; T^{i,0}_i] = (P^i \wedge)T^{i,0}_i$$

and therefore Eq. (13) is proven formally. The Lie-Poisson tensor $(P^i \wedge)$ is clearly skew-symmetric, such that the screw $(s_p \wedge s_t)$ is reciprocal to the screw $s_t$ and represents therefore a gyroscopic wrench which does not produce work. Furthermore, since $(s_p \wedge s_t)$ is a screw and $s_w$ is a screw, their sum is obviously still a screw which proves that the time derivative of the momentum screw is again a screw.

6 Conclusion

It has been shown that the derivatives of twists and wrenches, of any order, can be represented by screws, through their form that relates a point on the moving body to its instantaneous $n$-th derivative. The derivatives are pure screws, in case the lower-order derivatives are zero, Eq. (2) and Eq. (4).

It has also been shown that momentum has all properties of a screw, and that the dynamic equations of a rigid body can be concisely expressed in the screw form of Eq. (13), with added geometrical insight.

Last but not least, it has been shown that it is very meaningful for screw theorist to get acquainted to Lie Group theory, because it is not only analytically very useful, but also because there is such a large reservoir of proven results that just wait to be applied to the special case of rigid body motion.

References


