Visual Servoing For A Scale Model Autonomous Helicopter

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Abstract—A control design to stabilise a reduced scale autonomous helicopter equipped with a camera is presented. The proposed algorithm is motivated by recent work in image-based visual servoing control for under-actuated dynamic systems. This work is extended by considering models that contain weakly non-minimum phase zero dynamics. This is an important class of systems since an offset between the camera and the centre of mass for a typical ‘dynamic’ autonomous vehicle will result in zero dynamics occurring in the image dynamics. In this paper we propose a simplified model of dynamics of the helicopter and show that by placing some constraints in the choice of the position of the camera we can minimise the effect of the zero dynamics.

1 Introduction

Unmanned Air Vehicles (UAVs) are becoming an exciting new application area for modern non-linear control theory. Several authors in this last decade have contributed to the study and development of dynamics models which are dedicated to the UAV [10, 4]. Their small size, highly coupled dynamics and low cost implementation provide an ideal testing ground for sophisticated control techniques. One major problem that arises in the control of such vehicles is the difficulty of measuring non-inertial variables such as position, orientation and velocity. Acceleration and angular velocity may be measured using accelerometers and rate gyros. To overcome the problem of regulating the position of a UAV without expensive ground based systems, a number of authors have considered using visual feedback. This kind of control is expressed by ‘Visual Servoing’. Early implementations of visual servoing algorithms were used to position a robotic manipulator relative to a fixed visual target [1]. More recently, visual servoing has been proposed for applications in mobile robotics, for example in the control of a car-like vehicle [9, 6]. In all approaches feedback control is used to regulate an error function measured directly in the task space and derived from the control objective. This approach does not generalise well to the case where the dynamics of the system are important. Most existing applications exploit a high gain or feedback linearization (computed torque) design to reduce the system to a controllable kinematic model, for which the image-based visual servoing techniques were developed [3]. Recently Hamel and Mahony [2] presented a novel new algorithm for visual servoing of an under-actuated dynamic rigid body system based on exploiting the passivity-like properties of rigid body motion.

In this paper we consider a similar problem to that solved in [2]. However, the model considered is extended to incorporate the possibility of zero dynamics in the system resulting from an offset between the camera and the centre of mass of the UAV. In practice, this effect is unavoidable and, since the zero dynamics of a Hamiltonian system are Hamiltonian, may result in small non-decaying oscillations in the motion of the UAV. An idealized model of an autonomous model helicopter is considered as a base for the theoretical development undertaken. The principal contribution of the paper is to show that by choosing a suitable position for the camera, it is possible to limit the effects of the zero dynamics.

The paper is divided into five sections including the present overview. In Section 2 an idealised model of an autonomous helicopter is presented based on the Newton-Euler equations. In Section 3 the image dynamics are derived. Section 4 shows how to derive the proposed control law using a robust backstepping techniques. Finally, in the section 5, we give some simulations and results.

2 Helicopter Model

In this section a dynamic model of autonomous helicopter equipped with a camera is proposed. The model is derived from Newton-Euler equations by assuming that the helicopter body is rigid. For details of the dynamic equations, please refer to [4] or [7].

Denote the body airframe by letter A. Let \( I = \{E_x, E_y, E_z\} \) denote a right-hand inertial frame stationary with respect to the earth. Let the vectors \( \xi = (x, y, z)^T \) and \( \xi_e = (x_e, y_e, z_e)^T \) denote respectively the position of the centre of mass of the helicopter and the focal point of the camera relative to the frame \( I \). Let \( A = \{E_1, E_2, E_3\} \) be a (right-hand) body fixed frame for \( A \), and \( C = \{E_1, E_2, E_3\} \) denote the camera fixed frame of the camera, (see Fig. 1). In addition, let \( R \in SO(3) \) be an orthogonal rotation matrix which give the orientation of the helicopter relative to the inertial frame \( R : A \rightarrow I \). The orientation of the camera relative to the body fixed frame of
the helicopter is given by a fixed rotation $R_e : C \rightarrow A$.

![Figure 1: Components of lift forces and offsets.](image)

In the present paper, it is preferable to work directly with the force inputs decomposed into orthogonal components and assume that each of these inputs is an independent control. As shown on figure (1), the main rotor lift is decomposed into three components $(-u^2, w^1, -u) \in \mathcal{A}$ orientated in the directions $\{E_1^o, E_2^o, E_3^o\}$. Furthermore, the force due to the tail rotor is always oriented in the opposite direction to $E_2^o$ and is written $(0, -u^2, 0) \in \mathcal{A}$.

As shown in [8], by applying Newton’s equations of motions, the dynamics of an autonomous helicopter are

$$\dot{\mathbf{v}} = \mathbf{u}$$

$$m \ddot{x} = -m g e_3 - w^2 \mathbf{R}_e - (w_1 - w_3) \mathbf{R}_e$$

$$\ddot{\mathbf{R}} = R(k(\mathbf{J}))$$

$$\mathbf{I} \ddot{\mathbf{\Omega}} = -\mathbf{\Omega} \times \mathbf{\Omega} + (l^3_M w^1 - l^2_T w^3) e_1 + l^3_M w^2 e_2 + l^2_T w^3 e_3 + Q_M e_3 + Q_T e_2.$$  \hspace{1cm} (4)

In the above dynamics, the two torque controls governing roll and pitch, Eq. (2), generate sideways forces on the airframe which introduce zero dynamics into the system. These forces are known as small body forces.

In this model, it is assumed that $l_M = (0, 0, -l^3_M) \in \mathcal{A}$, where $l^3_M > 0$ and $l_T = (-l^3_T, 0, -l^2_T)$.

### 3 A Reduced Model

In this paper, it is proposed to find the optimal position for the camera that will reduced the effects of the small body forces. Assuming that the inertia matrix $\mathbf{I}$ is diagonal, then Eq.(4) may be rewritten explicitly as follows:

$$\mathbf{I}_1 \dot{\mathbf{\Omega}}_1 = -\Omega_3 \Omega_2 (\mathbf{I}_3 - \mathbf{I}_2) + (l^3_M w^1 - l^2_T w^3),$$

$$\mathbf{I}_2 \dot{\mathbf{\Omega}}_2 = -\Omega_3 \Omega_1 (\mathbf{I}_2 - \mathbf{I}_3) + Q_T + l^3_M w^2,$$

$$\mathbf{I}_3 \dot{\mathbf{\Omega}}_3 = -\Omega_2 \Omega_1 (\mathbf{I}_2 - \mathbf{I}_1) + Q_M + l^2_T w^3.$$  \hspace{1cm} (7)

It is assumed that the anti-torque associated with the rotor air resistance is known and differentiable. The control input $\mathbf{u}$ is decomposed into two parts

$$\mathbf{u} = \tilde{\mathbf{u}}_0 + \tilde{\mathbf{u}}$$

where $\tilde{\mathbf{u}}_0$ is the component of the torque input that counter-acts air resistance of the rotors and $\tilde{\mathbf{u}}$ is the part that contributes directly to the control of the helicopter. To counter-act the anti-torque $Q_M e_3$ due to the main rotor, choose

$$\dot{\mathbf{u}}_0^3 = -\frac{Q_M e_3}{I_T}.$$

Let $(I_2 - I_1) = \varepsilon$. Thus, one has

$$\mathbf{I}_3 \dot{\mathbf{\Omega}}_3 = -\Omega_2 \Omega_1 \varepsilon + l^2_T \ddot{\mathbf{\Omega}}^3.$$

Assume that $\Omega_3(0) = 0$ (otherwise initially stabilise the yaw before commencing the control design) then by choosing

$$\ddot{\mathbf{\Omega}}_3^1 = \frac{l^3_M}{I_T} \ddot{\mathbf{\Omega}}_3^1 = -\frac{l^3_T}{I_T} Q_M,$$

one ensures that $\Omega_3^1 \equiv 0$ for all time.

Choosing $\ddot{\mathbf{\Omega}}_3^1 = -\frac{Q_M}{I_T}$ cancels the effect of the anti-torque $Q_M(t)$. It is also necessary to cancel the contribution to the first component torque due to the input $\ddot{\mathbf{\Omega}}_3^3$ and non-zero offset $\dot{\mathbf{\Omega}}_3^3$. Set

$$\ddot{\mathbf{\Omega}}_3^1 = \frac{l^3_M}{I_T} \ddot{\mathbf{\Omega}}_3^1 = -\frac{l^3_T}{I_T} Q_M,$$

Based on the above choices, the equation of motion Eqn's 2-4 may be written in a reduced form:

$$\dot{\mathbf{u}} = -u \mathbf{R}_e - \tilde{\mathbf{u}}^2 \mathbf{R}_e + ((\tilde{\mathbf{u}}^1 - \tilde{\mathbf{u}}^3) \mathbf{R}_e + mg e_3 + \frac{l^3_M}{I_T} - \frac{l^3_T}{I_T} Q_M \mathbf{R}_e + Q_T \mathbf{R}_e)$$

$$\mathbf{I} \ddot{\mathbf{\Omega}} = (l^3_M \ddot{\mathbf{\Omega}} + l^3_T \ddot{\mathbf{\Omega}}^3) e_1 + l^3_M \ddot{\mathbf{\Omega}} e_2.$$  \hspace{1cm} (11)

The inputs for this system are $u$, $\tilde{\mathbf{u}}^1$ and $\tilde{\mathbf{u}}^3$. In addition, the control input $\ddot{\mathbf{\Omega}}^2$ is used to stabilize $\Omega_3(t)$ at 0. All the terms in Eq. (10) directly independent of the control input are written separately in brackets. These terms form the disturbance to the translation dynamics, dominate by the gravitational force $mg$, but also depending on the rotation matrix $R$ as well as the rotor drag terms. A consequence of the perturbation terms in Eq (10) is that the stationary point of the dynamics Eqn’s (1,10, 3 and 11) corresponding to the hover point requires a non-trivial rotation $R$. Indeed, in stationary flight any helicopter must fly with slight permanent inclination of the principal lift force $-u \mathbf{R}_e$ to compensate for the small body force resulting from the torque control necessary to cancel rotor drag.

It is important for control design in visual servoing to represent the dynamics of the helicopter in the camera fixed frame. Let $v_e$ be the velocity of focal point of the camera relative to the inertial frame, and let $V_e$ its representation in the camera fixed frame.
Let $d \in \mathbb{R}^3$ be a translation vector between $\mathcal{A}$ and $\mathcal{C}$, one has

$$\xi_c = \xi + Rd$$

(12)

Considering only a displacement in the $E^2_3$ axis, that is $d = d^c e_3 = (0,0,d^c)^T$, the rotational matrix $R_c = I_3$, the full linear dynamics of a helicopter expressed in the camera fixed frame are given by

$$\ddot{\xi}_c = RV_c,$$
$$m\ddot{V}_c = -m\Omega \times (V_c - \Omega \times d) - we_3$$

(13)

$$+mgR^T e_3 + \left( \frac{\beta^3 - \beta^3_T}{\beta^3_M - \beta^3_T} \right) \dot{Q}Mc_2 + \frac{Q_T}{\beta^3_M} I e_1$$

$$+ \left( -\ddot{w}^3 + \frac{m}{\beta^3_M} \ddot{w}^3 + \left( \ddot{w}^1 - \ddot{w}^3 \right) - m \frac{\beta^3}{\beta^3_M} \ddot{w}^3 \right) e_2$$

(14)

From this equation it is clear that the displacement $d^c$ may be chosen such that the first entries of the right hand side of Eq. (15) are independent of the control action $\ddot{w}^2$. Set:

$$d^c = \frac{I_2}{m\beta^3_M}$$

(15)

In order, to cancel the centrifugal forces $\Omega \times (\Omega \times d)$ introduced by the offset of the point $\xi_c$, define the following modified control $\ddot{u}$:

$$\ddot{u} = u + m\ddot{d} \left( (\Omega_1)^2 + (\Omega_2)^2 \right)$$

(16)

By this choice and using Eq.(15), the reduced linear model dynamics may be written:

$$\ddot{\xi}_c = RV_c,$$
$$m\ddot{V}_c = -m\Omega \times (V_c - \ddot{\omega} e_3 + mgR^T e_3 + X \left( -\ddot{w}^1 + \frac{\beta^3 I_2 - \beta^3 M I_1}{\beta^3_M} \ddot{w}^3 \right) e_2$$

(17)

(18)

where $X = \left( \frac{\beta^3 - \beta^3_T}{\beta^3_M - \beta^3_T} \right) \dot{Q}Mc_2 + \frac{Q_T}{\beta^3_M} I e_1$ and we assume that the control algorithm has access to $\dot{X}$ and $\ddot{X}$.

In Eq. (18), the presence of the small body forces, which couple torque inputs to the translational dynamics are only present in the second axis direction $E^2_3$. The zero dynamics are generated by the additional control inputs $\ddot{w}^1$ and $\ddot{w}^3$ that are not exactly cancelled by the choice of camera position. In practice, the presence of the small body forces destroys the design of a robust control, in particular the backstepping procedure which use to give our design. In the present paper and as proposed in [8, 4], to avoid this problem, we ignore the control terms contributing to the zero dynamics effect. Furthermore, as proposed in [8], we can give an analysis which shows that as long as the small body forces are sufficiently small the proposed control design in this paper will ensure the same properties for the full system.

4 Image Dynamics

Let a set of $n$ points $P_i$ of a fixed target, relative to the inertial frame $\mathcal{I}$ and observed from the camera, and Let $P_i$ the representation of $P_i^* \in \mathcal{C}$ in the camera fixed frame, then $P_i$ is given by

$$P_i = R^T (P_i^* - \xi_c)$$

A backstepping control design has passivity-like properties from virtual input to the backstepping error [5]. In a recent paper [2] it was shown that these structural passivity-like properties are present in the image space dynamics if and only if the spherical projection of an observed point is used. Denoting the spherical projection of an image point $P_i$ by $p_i$ the image dynamics are given by (see [2] for more details)

$$\dot{p}_i = -sk(\Omega)c_i + \pi_p \frac{V_i}{r_i}$$

(19)

Here $r_i = \|P_i\|$ and $V_i \in \mathcal{C}$ is the observed velocity of the target point represented in the body fixed frame of the camera. The matrix $\pi_p = (I_3 - p_i p_i^T)$ is the projection onto the tangent space of the spherical image surface at the point $p_i$.

If the inertial velocity of the target point $P_i^*$ is equal to zero then $\dot{V}_i = -V_c$, where $V_c$ is the velocity of the camera relative to the camera fixed frame. Thus Eq. (19) becomes:

$$\dot{p}_i = -\Omega \times p_i - \frac{\pi_p}{r_i} V_c$$

Here the focal length of the spherical camera has been normalized to unity and it is assumed that the target points are stationary in the inertial frame.

We wish to stabilize the camera to a desired pose (position and orientation) with respect to a target of interest. Let $p_i^*$ denote the desired visual features, fixed relative to the inertial frame. The image based error vector chosen is

$$\delta = \text{vect}(p_i - p_i^*) \in \mathbb{R}^{3n}$$

Let

$$\Pi = \left( \begin{array}{c} \frac{1}{r_i} (J_3 - p_i p_i^T) \\ \vdots \\ \frac{1}{r_m} (J_3 - p_m p_m^T) \end{array} \right)$$

Computing the first order dynamics of $\delta$, it yields

$$\dot{\delta} = -\text{diag}(sk(\Omega)) \delta - \Pi V_c$$

A common approach taken in image based visual servoing is to average the available visual information contained in the full error $\delta$. This is achieved by introducing a combination matrix $C$ that ‘codes’ the averaging process and defining a reduced error

$$\delta_e = C \delta, \quad C \in \mathbb{R}^{3 \times 3n}$$

(20)
Following the approach of Hamel and Mahony [2], the reduced error is only three dimensional and it is impossible to derive full pose information from the reduced error. However, the reduced error does suffice to fully specify UAVs position and yields passivity-like behaviour suitable for application of backstepping control design as long as the matrix $C$ satisfies

$$CII = Q > 0 \text{ and } C_{\text{diag}}(-sk(\Omega)) = -sk(\Omega)C, \forall \Omega \in \mathbb{R}^3.$$ 

A matrix respecting these conditions will be of the form $C_{\alpha} = [ \alpha_1 I \cdots \alpha_n I ]$, $\alpha_i \geq 0, i = 1, ..., n$. Note that the property $C_{\alpha}P = Q_{\alpha} > 0$ relies on the fact that $r_1 > 0$.

The reduced error vector considered is $\delta_1 = C_{\alpha}\delta$ and the full dynamics of $\delta_1$ are given by

$$\dot{\delta}_1 = -sk(\Omega)\delta_1 - Q_\alpha V_c.$$ \hspace{1cm} (21)

The exact value of $Q_\alpha$ is unknown, however, it is possible to obtain bounds on the eigenvalues of $Q_\alpha$ from upper and lower bounds on distance of the UAV to the target [2] and these are used in the control design.

5 Visual servoing for helicopter

In this section, a control design robust backstepping techniques [5] is proposed for visual servoing of the idealised autonomous helicopter dynamics proposed earlier.

From Eq. (21) and the reduced rigid body dynamics Eqn’s (18, 3 and 11), the full dynamics of the error $\delta_1$ may be written:

$$\dot{\delta}_1 = -sk(\Omega)\delta_1 - Q_\alpha V_c \hspace{1cm} (22)$$

$$m\dot{V}_c = -m\Omega \times V_c - \bar{\omega}_3 + mgR^T e_3 + X \hspace{1cm} (23)$$

$$\dot{R} = Rsk(\Omega) \hspace{1cm} (24)$$

$$I_\Omega = (I_{M}^3 \bar{\omega}^1 - I_{M}^3 \bar{\omega}^3)\epsilon_1 + I_{M}^3 \bar{\omega}^2 e_2 \hspace{1cm} (25)$$

Define the first storage function $S_1$ by:

$$S_1 = \frac{1}{2} |\delta_1|^2 \hspace{1cm} (26)$$

Deriving $S_1$ and recalling Eq. (22) yields:

$$\dot{S}_1 = -\delta_1^T Q_\alpha \dot{\delta}_1 \hspace{1cm} (27)$$

If the velocity $V_c$ are available as a control input, then the choice of $V_c = \delta$ is sufficient to stabilise $S_1$. Thus the virtual control chosen for this step is

$$V_c = \frac{k_1}{m} \dot{\delta}_1 \hspace{1cm} (28)$$

where $k_1$ is a positive constant. Since the matrix $Q_\alpha$ is positive definite and if $V_c^\ast \equiv V_c$, then $\dot{S}_1 = -\frac{k_1}{m} \delta_1^T Q_\alpha \delta_1$ is negative definite in $\delta_1$. Thus, Eq.(22) becomes

$$\dot{\delta}_1 = -sk(\Omega)\delta_1 - \frac{k_1}{m} Q_\alpha \delta_1 - \frac{k_1}{m} Q_\alpha \delta_2, \hspace{1cm} (29)$$

where $\delta_2$ is a new error term given by

$$\delta_2 = \frac{m}{k_1} V_c - \delta_1, \hspace{1cm} (29)$$

the difference between the true and the desired velocity. The derivative of $S_1$ may be written

$$\dot{S}_1 = -\frac{k_1}{m} \delta_1^T Q_\alpha \delta_1 - \frac{k_1}{m} \delta_2^T Q_\alpha \delta_2 \hspace{1cm} (30)$$

The second storage function considered is

$$S_2 = S_1 + \frac{1}{2} \delta_2^T \delta_2 \hspace{1cm} (31)$$

Taking the derivative of $S_2$ and substituting for Eq. (23), yields

$$\dot{S}_2 = -\frac{k_1}{m} \delta_1^T Q_\alpha \delta_1 + \frac{k_1}{m} \delta_2^T Q_\alpha \delta_2 + \frac{k_1}{m} \delta_3 \hspace{1cm} (32)$$

Let $(-\bar{\omega}_3 + mgR^T e_3)^T$ the virtual control chosen for this step $(-\bar{\omega}_3 + mgR^T e_3)^T = -\frac{k_1}{m} \delta_2 - X$. With this choice the derivative of $\delta_2$ is given by

$$\dot{\delta}_2 = -sk(\Omega)\delta_2 + \frac{k_1}{m} Q_\alpha \delta_1 - \frac{k_1}{m} (k_2 I_3 - Q_\alpha) \delta_2 + \frac{k_1}{m} \delta_3 \hspace{1cm} (33)$$

where $\delta_3$ represents the third error term

$$\delta_3 = \frac{m}{k_1^3} \left( (-\bar{\omega}_3 + mgR^T e_3) + \frac{k_2}{m} \delta_2 + X \right) \hspace{1cm} (34)$$

In the formal process of backstepping the new error $\delta_3$ would now be differentiated. This requires a formal time derivative of the input $\bar{u}$. To provide this the input $\bar{u}$ is dynamically extended

$$\ddot{\bar{u}} = \ddot{u} \hspace{1cm} (35)$$

The motivation for adding a double integrator here is to ensure that the relative degree of the new input $\bar{u}$ with respect to the position co-ordinates $\xi$ is the same as the relative degree of $\bar{u}^1, \bar{u}^2$ and $\bar{u}^3$ with respect to the position. In the present derivation, all inputs are chosen to have relative degree four with respect to the position $\xi$.

The derivative of the second storage function is:

$$\dot{S}_2 = -\frac{k_1}{m} \delta_1^T Q_\alpha \delta_1 - \frac{k_1}{m} \delta_2^T (k_2 I_3 - Q_\alpha) \delta_2 + \frac{k_1}{m} \delta_3 \hspace{1cm} (36)$$

where $k_2$ is chosen such that $k_2 > \lambda_{\max}(Q_\alpha)$. The backstepping process continued by taking the derivative of $\delta_3$

$$\dot{\delta}_3 = -sk(\Omega)\delta_3 + \frac{k_1}{m} Q_\alpha \delta_1 - \frac{k_1}{m} (k_2 I_3 - Q_\alpha) \delta_2 + \frac{k_1}{m} \delta_3$$

$$+ \frac{m}{k_1^3} \bar{X} - \frac{m}{k_1^3} \left( \bar{\omega}_3 + sk(\Omega)(\bar{\omega}_3 - X) \right) \hspace{1cm} (37)$$

where $\bar{X}$ is chosen such that $\bar{X} > \lambda_{\max}(Q_\alpha)$. The backstepping process is continued by taking the derivative of $\delta_3$
The virtual control chosen in this step is
\[
\hat{\dot{e}}_3 \equiv \frac{k_1^2 k_2 (k_3 + k_3 k_1)}{m^2} \delta_3 + \hat{\dot{X}}.
\]
Using this choice yields
\[
\dot{\delta}_3 = -sk(\Omega)\delta_3 + \frac{k_1}{m} Q_\alpha \delta_1 - \frac{k_1}{m} (k_2 I_3 - Q_\alpha) \delta_2 - \frac{k_3}{m} \hat{\dot{X}} - \frac{(k_3 + k_1 k_1)}{m} \delta_3 - \frac{(k_3 + k_1 k_1)}{m} \delta_4.
\]

Here \(\delta_4\) is the final error term introduced in the design procedure
\[
\dot{\delta}_4 = \frac{m^2 (\hat{\dot{e}}_3 + sk(\Omega)(\hat{\dot{e}}_3 - X) - \hat{\dot{X}})}{k_1^2 k_2 (k_3 + k_3 k_1)} - \delta_3.
\]
Let \(S_3\) be a storage function associated with this step of the procedure
\[
S_3 = S_2 + \frac{1}{2} \delta_4^2
\]
Deriving \(S_3\) and recalling Eq. (34) one obtains
\[
\dot{S}_3 = -\frac{k_1}{m} \delta_3^T Q_\alpha \delta_1 - \frac{k_1}{m} \delta_2^T (k_2 I_3 - Q_\alpha) \delta_2 + \frac{k_1}{m} \delta_4^T Q_\alpha \delta_1
\]
\[
+ \frac{k_3}{m} \delta_4 Q_\alpha \delta_2 - \frac{(k_3 + k_3 k_1)}{m} \delta_3 - \frac{k_3}{m} \delta_3^T \delta_3.
\]

The derivative of \(\delta_4\) is
\[
\dot{\delta}_4 = sk(\Omega) \delta_3 - \frac{k_1}{m} Q_\alpha \delta_1 + \frac{k_1}{m} (k_2 I_3 - Q_\alpha) \delta_2
\]
\[
+ \frac{k_1}{m} \delta_3 + \frac{k_1}{m} (k_3 + k_1 k_1) \delta_4 - \frac{m^2}{k_1^2 k_2 (k_3 + k_3 k_1)} \hat{\dot{X}}
\]
\[
+ \frac{m^2}{k_1^2 k_2 (k_3 + k_3 k_1)} (\hat{\dot{e}}_3 + sk(\Omega)(\hat{\dot{e}}_3 - X))
\]
\[
+ \frac{m^2}{k_1^2 k_2 (k_3 + k_3 k_1)} sk(\Omega)(\hat{\dot{e}}_3 - \hat{\dot{X}})
\]
At this stage the actual control inputs enter into the equations through \(\hat{\bar{u}} = \bar{u}\) and \(\Omega\) via Eq. (25).
The final stage of the backstepping process is choose a control input. The following vector equation is solved for the control inputs \(\hat{u}\) and \(\hat{w}\).
\[
\left(-\hat{\dot{e}}_3 + \Omega \times (-\hat{\dot{e}}_3 + X)\right) = -sk(\Omega)(\hat{\dot{e}}_3 - X) - \hat{\dot{X}} + \frac{k_1^2 k_2 (k_3 + k_2 k_1)}{m^2} (-sk(\Omega) \delta_3 - \frac{k_2 k_1 + k_3 + k_4}{m} \delta_4
\]
\[
+ \frac{k_0 k_1}{m} \delta_3)
\]
Given the above choice the dynamics of \(\delta_4\) becomes
\[
\dot{\delta}_4 = -\frac{k_1}{m} Q_\alpha \delta_1 + \frac{k_1}{m} (k_2 I_3 - Q_\alpha) \delta_2 + \frac{(k_2 k_1 + k_3 + k_4)}{m} \delta_3
\]
\[
- \frac{k_4}{m} \delta_4
\]
The last storage function in the backstepping procedure is the Lyapunov candidate function
\[
S_4 = S_3 + \frac{1}{2} \delta_4^2
\]
Deriving \(S_4\) and recalling Eq.(38) one obtains
\[
\dot{S}_4 = \frac{k_1}{m} \delta_3^T Q_\alpha \delta_1 - \frac{k_1}{m} \delta_3^T (k_2 I_3 - Q_\alpha) \delta_2 + \frac{k_1}{m} \delta_3^T Q_\alpha \delta_1
\]
\[
+ \frac{k_1}{m^2} (k_2 I_3 - Q_\alpha) \delta_2 - \frac{k_1}{m} \delta_4^T Q_\alpha \delta_1
\]
Recall that the dynamics of the angular velocity of the camera expressed in the body fixed frame are given by Eq. (11). Expanding the vector cross product and collecting terms then the left hand side of Eq. (39) may be written as:
\[
\begin{pmatrix}
\dot{\bar{w}}_3 \bar{w}_3 T \bar{u}
\end{pmatrix}
\]
\[
\bar{u} = f_3 \bar{w}_3 T X + \frac{\bar{w}_3}{f_3} \bar{w}_3 T X
\]
Thus, as long as \(\bar{u} \neq 0\) there is a unique control input \((\bar{u}, \bar{w}_1, \bar{w}_2, \bar{w}_3)\) that solves Eq. (39). In hover conditions, the condition \(\bar{u} \neq 0\) will always hold since \(\bar{u}\) corresponds to the heave force used to compensate gravitational acceleration. A second interesting observation is that the presence of the drag terms \(X\) introduce parasitic terms \(sk(\Omega) X\) which are cancelled by the input \(\bar{u}\). Finally, as indicated in the subsection 2.1, the control design is complete when the the third input \(\bar{w}_3\) free to fix \(\Omega^3\) at zero.

**Proposition [2]**
Consider the dynamics given by Eqn’s (22-25). Let the vector controller given by Eq.(39) and recover the original inputs from Eq.(41). Then the main objective \(\delta_1\) converges to zero if the control gains satisfy the following constraints:
\[
k_1 > 0
\]
\[
k_2 > \lambda_{\text{max}} (Q_\alpha)
\]
\[
k_3 > k_1 \lambda_{\text{max}} (Q_\alpha) \left( \frac{1}{\lambda_{\text{min}} (Q_\alpha)} + \frac{1}{k_2 - \lambda_{\text{min}} (Q_\alpha)} \right)
\]
\[
k_4 > k_1 \left( \frac{\lambda_{\text{max}} (Q_\alpha)}{\lambda_{\text{min}} (Q_\alpha)} + \frac{k_2 - \lambda_{\text{min}} (Q_\alpha)}{k_2 - \lambda_{\text{max}} (Q_\alpha)} \right)
\]
where \(\lambda_{\text{min}} (Q_\alpha)\) and \(\lambda_{\text{max}} (Q_\alpha)\) represent the lower and upper bounds of \(Q_\alpha\).
The proof of this proposition is a direct application of the principles of the backstepping based on the development leading up the proposition statement.

### 6 Simulation and results
In this section we present a simulation with the aim to validate the proposed control design. The task considered is to position a camera relative to the planar square target. The target is modelled by four
points on the vertices of the square. Naturally for this task, the signals available are the pixel co-ordinates of the four points observed by the camera, denoted \{((u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4))\}. The simulated spherical co-ordinates \((x_i, y_i, z_i)\) of the four points are respectively given by \(x_i = \frac{u_i}{\sqrt{u_i^2 + v_i^2}}, y_i = \frac{v_i}{\sqrt{u_i^2 + v_i^2}}\) and \(z_i = \sqrt{1 - u_i^2 - v_i^2}\). For more details the reader is referred to [2].

The desired image feature are chosen such that the camera is located several meters above the square. It is defined by : \{(-a, a), (a, a), (-a, -a), (a, -a)\} where \(a\) represents the ratio between the vertex length and the final desired range. In this simulation the parameter \(a\) has been chosen equal to 0.4851 corresponding to a location of six meters of the camera above the target.

The magnitude of the initial force input is chosen to be \(u_0 = \gamma m \approx 177\) corresponding to the fact that the helicopter is initially in hover flight. The initial position is:
\[
\xi_0 = \begin{pmatrix} 5 & 4 & -16 \end{pmatrix}^T, \quad \dot{\xi}_0 = 0 \quad \text{and} \quad \phi_0 = \dot{\phi}_0 = 0.
\]

The center of the target \(\hat{\xi}\) is chosen to be:
\[
\hat{\xi}_0 = \begin{pmatrix} 0 & 0 & -6 \end{pmatrix}^T
\]

the choice of the values of \(\lambda_{\text{min}}(Q_a)\) and \(\lambda_{\text{max}}(Q_a)\) depends respectively on initial conditions and the desired location, it follows that:

\[
\lambda_{\text{min}}(Q_a) = 0.0206 \quad \text{and} \quad \lambda_{\text{max}}(Q_a) = 0.3686.
\]

From the above choice, we have used the following control gains: \(k_1 = 0.2, k_2 = 1, k_3 = 3, k_4 = 3\), that satisfies conditions given in Proposition.

Simulations results of the helicopter system control algorithm are given on figures 2 and 3.

**References**


