Rigid Body Analysis of Power Grasps: Bounds of the Indeterminate Grasp Force

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Abstract

The equilibrium equation of a power grasp cannot determine the grasp force uniquely in general. This paper proves that the indeterminate grasp force is within a bounded hyper-polyhedron not necessarily convex by considering the compatibility with rigid body motion. The maximum friction angle is found at its vertex. Thus, if all forces at the vertices are within the friction cone, then all forces in the hyper-polyhedron are within the friction cone. Based on this analysis, we present an algorithm for computing joint torques that can balance a given set of external disturbances.

1 Introduction

Power grasping or enveloping grasping enables a multifingered hand to grasp an object firmly by constraining it by multiple contacts on the surfaces of the hand. The overconstrained nature of power grasping has been attracting research interest. It can balance a set of external disturbances without changing joint torques as long as slip does not occur[6]. Stability of power grasping is analyzed in terms of a stiffness matrix[7] and a linearized dynamic equation[3]. Kinematic issues are discussed in [4] and planning a power grasp is discussed in [8].

One of the fundamental problems is that the equilibrium equation of a power grasp cannot determine the grasp force uniquely in general. One way to avoid this indeterminacy is to introduce a compliant contact model[1, 2, 5]. However, identification of the compliance parameters of the contact model is necessary. The motivation of this study is that we wish to compute joint torques of a power grasp only using the equilibrium equation without determining the grasp force uniquely.

We showed that some solutions to the equilibrium equation can be excluded by considering the compatibility with the laws of rigid body motion[10]. Over-constraining an object in power grasps results in in-feasible sliding directions of the contacts. A static friction force acts only in the opposite direction of the tendency of slip. Friction forces in the opposite directions of the infeasible sliding do not exist.

This paper further analyzes the indeterminate solutions to the equilibrium equation and generalizes the previous results in [10]. We prove that they are within a bounded hyper-polyhedron not necessarily convex. The maximum friction angle is found at its vertex. Therefore if all forces at the vertices are within the friction cone, then all forces in the hyper-polyhedron are within the friction cone. This is a sufficient condition for the contacts not to slip. Based on this analysis, we present an algorithm for computing joint torques that can balance a given set of external disturbances.

This paper is organized as follows. Section 2 presents our previous results slightly refining the condition that sliding in power grasps is incompatible with rigid body motion. Section 3 proves that the indeterminate grasp force in a power grasp is within a bounded hyper-polyhedron. Section 4 shows examples. Section 5 presents an algorithm for computing joint torques of a power grasp that can balance a given set of external disturbances.

2 Indeterminate grasp force

2.1 Equilibrium equation

Fig. 1 shows a power grasp with a multifingered hand. The notation for the grasp is as follows.

$N$: number of fingers
$M_i$: number of contact points on the $i$th finger
$L_i$: total number of contact points ($= \sum_{i=1}^{N} M_i$)
$L$: total number of joints ($= \sum_{i=1}^{N} L_i$)
$C_{ij}$: $j$th contact point on the $i$th finger
$c_{ij}$: position vector from the object frame to $C_{ij}$
$f_{ij}$: contact force at $C_{ij} \in \mathbb{R}^3$ ($\mathbb{R}^2$)
$f_i = (f_{i1}^T \cdots f_{iM_i}^T)^T$
Throughout this paper, \( v \theta \) and \( \theta \) denote the velocities of the object, respectively.

The relationship between the joint torque \( \tau \), the joint velocity of the \( i \)th joint of the \( i \)th finger, and the \( j \)th joint of the \( i \)th finger is given by

\[
\tau_{ij} = \dot{\theta}_{ij} : \text{joint torque of the } j\text{th joint of the } i\text{th finger}
\]

\[
\dot{\theta}_{ij} = \left( \dot{\theta}_{i1} \cdots \dot{\theta}_{iL_i} \right)^T \in R^{L_i}
\]

where

\[
\dot{\theta}_{ij} : \text{joint velocity of the } j\text{th joint of the } i\text{th finger}
\]

\[
\tau_{ij} : = \left( \tau_{i1} \cdots \tau_{iL_i} \right)^T \in R^{L_i}
\]

\[
\dot{\theta}_{ij} : = \left( \dot{\theta}_{1j} \cdots \dot{\theta}_{L_jj} \right)^T \in R^{L_j}
\]

\[
v_0 \in R^3 (R^2), \omega_0 \in R^3 (R) : \text{linear and angular velocities of the object, respectively}
\]

\[
w_{ext} \in R^6 (R^3) : \text{external wrench}
\]

The dimension in a parenthesis is for a planar grasp throughout this paper.

When an external wrench \( w_{ext} \) is exerted on the object, the equilibrium equation of the object is

\[
W f + w_{ext} = 0
\]

where \( W \in R^{6 \times 3M} (R^{3 \times 2M}) \) is the wrench matrix given by

\[
W = \left( \begin{array}{c} W_1 \cdots W_N \end{array} \right) \in R^{6 \times 3M} (R^{3 \times 2M})
\]

\[
W_i = \left( \begin{array}{c} I \cdots I \\ S(c_{i1}) \cdots S(c_{iL_i}) \end{array} \right) R^{6 \times 3M_i} (R^{3 \times 2M_i})
\]

\[
S(a) \text{ is a } 3 \times 3 \text{ skew symmetric matrix, i.e. } S(a)b = a \times b, \forall a, b \in R^3.
\]

The dimension in a parenthesis is for a planar grasp.

The relationship between the joint torque \( \tau \) and the grasp force \( f \) is given by

\[
J^T f = \tau
\]

where

\[
J = \left( \begin{array}{c} J_1 \cdots 0 \\ \vdots \vdots \vdots \\ 0 \cdots J_N \end{array} \right) \in R^{M \times L} (R^{2M \times L})
\]

and \( J_i \in R^{M_i \times L_i} (R^{2M_i \times L_i}) \) is the jacobian matrix between the multiple contact points and the joints of the \( i \)th finger.

From Eqs. (1) and (4) we have

\[
Af = \left( \begin{array}{c} -w_{ext} \\ \tau \end{array} \right)
\]

where

\[
A = \left( \begin{array}{c} W \\ J^T \end{array} \right) \in R^{(6+L) \times 3M} (R^{(3+L) \times 2M}).
\]

In power grasps, \( \text{Ker}A \) exists in general. Let \( n = \text{dimKer}A \) and \( H \in R^{3M \times n} (R^{2M \times n}) \) be a matrix whose columns span the bases of \( \text{Ker}A \), that is, \( \text{Im}H = \text{Ker}A \). The general solution to Eq. (6) is

\[
f = v + Hx
\]

where \( v \) is a particular solution. Due to the homogeneous solution \( h = Hx \), the general solution \( f \) is not unique.

Fig. 2 shows an example of a two-fingered power grasp [10] in which \( \text{dim} \text{Ker}A = 1 \). Figs. 2(a) and (b) show the homogeneous solution \( h = Hx \) and its negative \( -h \), respectively.

2.2 Constraint on sliding directions and grasp forces

In Fig. 2, the contact points all together cannot slide in the same directions as the tangential (friction) components of \( h \) as shown in Fig. 2(c). Nor can they
all together in the opposite directions as shown in Fig. 2(d). When $\text{Ker}A$ exists, that is, when the solution to the equilibrium equation (6) is indeterminate, the sliding directions of the contact points are constrained. This section summarizes the results in [10].

The constraint that the contact points do not break but can slide is

$$A^T \begin{pmatrix} V \\ \dot{\theta} \end{pmatrix} = T\dot{Y}$$

(9)

where $V = (\begin{pmatrix} v_i^T \\ \omega_i^T \end{pmatrix})^T$, $T \in R^{3M \times 2M}$ ($R^{2M \times M}$) is the matrix whose columns are the tangent unit vectors at the contact points, and $\dot{Y} \in R^{2M}$ ($R^M$) is the vector whose elements are sliding velocities at them.

The friction considered in this paper is “static”. Although this paper is concerned with sliding of a subject actually, but to see whether or not static friction forces are compatible with rigid body motion.

This section shows that IGF is within a bounded hyper-polyhedron and presents an algorithm for computing its vertices. We do not consider a static friction limit in this section.

Let $H_i$ be the $i$th row of $H = TT^H$. Each element $f_{ii} = v_i + H_{ii}x$, $i = 1, \ldots, K$ is a function of $x \in R^n$. The set of $x$ which satisfies $f_{ii} = 0$ is a hyper-plane in $R^n$. It divides $R^n$ into two regions. In one region, $f_{ii} > 0$ and in the other, $f_{ii} < 0$. The set of the $K$ planes $f_{ii} = v_i + H_{ii}x = 0$, $i = 1, \ldots, K$ divides $R^n$ into at most $2^K$ regions.

Consider the next simple example.

$$f_t = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(17)

This example does not correspond to a grasp but is useful to examine the region of IGF.
For any condition in Eq. (15) is now given by the non-zero element $S_i$. Theorem 2: A necessary and sufficient condition that $S_i$ is a subregion of IGF is that it is bounded. Proof) First we prove that IGF is bounded. From the assumption that $\text{Ker } H_t = \emptyset$, any $h_t$ has at least one non-zero element $h_{0i}$. Set $z = h$ in Eq. (15). Let $h_0 = h/\|h\|$. Then $h_0 \in \text{Im } H$ and $\|h_0\| = 1$. The condition in Eq. (15) is now given by

$$\text{for all } i \in I, \quad (v_{ti} + h_i h_{0ti}) h_{0ti} > 0$$

(18)

where $h_{0ti} = T^T h_0 = (h_{0ti})$. When the sign of $v_{ti}$ is different from that of $h_{0ti}$, if $\|h\| > -v_{ti}/h_{0ti}$, then $(v_{ti} + h_i h_{0ti}) h_{0ti} > 0$. Therefore if

$$\|h\| > \max_{i \in I}(-v_{ti}/h_{0ti}),$$

(19)

Eq. (18) holds. $\|h\|$ is bounded so that Eq. (18) does not hold.

To prove that a bounded subregion is IGF, we prove that NGF is unbounded. If $f_i = v_{ti} + h_{ti}$ is NGF, then there exists a homogeneous solution $z_i$ such that

$$\text{for all } i \in I, \quad (v_{ti} + h_{ti}) z_{ti} > 0.$$  

(20)

For any $\lambda > 0$

$$\text{for all } i \in I, \quad (v_{ti} + h_{ti} + \lambda z_{ti}) z_{ti} > 0.$$  

(21)

Therefore $f_i' = v_{ti} + h_t + \lambda z_{ti}$ is NGF. The signs of its elements are same as those of $f_i$. Since $\lambda > 0$ is arbitrary, $f_i'$ is unbounded (QED).

In the example in Fig. 3, $S_1 \cup S_2 \cup S_3$ is the region of IGF. In general, it is a bounded hyper-polyhedron not necessarily convex. It is denoted by $S$.

3.2 Vertices of $S$

An important geometric feature of $S$ is a vertex. As we will show in section 3.3, the maximum friction angle is found in the vertices of $S$. We present an algorithm for computing the vertices of $S$.

The set of $x$ such that $f_i = v_{ti} + H_t x = 0$ defines a plane. There are $K$ planes ($i = 1, \ldots, K$). When $n$ planes out of them are not parallel to each other, they intersect at a point. In Fig. 3, there are six intersections $V_1$ through $V_6$. $V_1$, $V_2$, $V_3$, and $V_5$ are the vertices of $S$, but $V_4$ and $V_6$ are on the boundary of $S = S_1 \cup S_2 \cup S_3$. In general we have

Theorem 3: The intersections include all vertices of $S$. They may include a point which is not a vertex of $S$ but on the boundary of $S$ or in $S$.

Proof) Assume that all subregions adjacent to an intersection are NGF. Since each subregion is convex, no subregion contains others inside it. Therefore the whole space $R^n$ is NGF. This does not hold. As we show in Appendix A, if $\text{Ker } H_t = \emptyset$, there exists a particular solution $v$ such that the friction component $v_t$ is orthogonal to any $h_t = H_t x$. From Theorem 1, it is IGF. Therefore at least one subregion is IGF. The intersection is a vertex of $S$ or on the boundary of $S$.

When all subregions adjacent to the intersection are IGF, the intersection exists in $S$ (QED).

Although some of the intersections are not a vertex, this does not cause a serious problem since they are in $S$. Using Theorem 3, the vertices of $S$ (including internal points of $S$) can be computed by the following procedure.

Let $P_k \in R^{n \times K}$ ($n < K$) be a matrix which selects $n$ rows from $K$ rows. There are $\nu C_n$ such matrices and the subscript $k$ ranges from 1 to $\nu C_n$. Out of $K$ planes, $n$ planes are selected by

$$P_k f_t = P_k v_{ti} + P_k H_t x = 0.$$  

(22)

For them to intersect at a point, $\text{rank } (P_k H_t) = n$. For $k = 1, \ldots, \nu C_n$, if $\text{rank } (P_k H_t) = n$, compute

$$x = -(P_k H_t)^{-1} P_k v_{ti}. $$

(23)

The corresponding grasp force is

$$u_k = v - (P_k H_t)^{-1} P_k T^T v.$$  

(24)

We readily have the follows.
fact that a friction cone is convex. Let 

\[ \theta = \max(\theta_1, \theta_2) \] 

where \( \theta \) contains \( \theta_1 \) and \( \theta_2 \). The friction cone with the friction angle \( \theta \) contains \( \theta_1 \) and \( \theta_2 \). Since it is convex, the forces \( c = (1-s)c_1 + sc_2, \quad 0 \leq s \leq 1 \) are inside the friction cone. Therefore the friction angle of \( \theta \) does not exceed \( \theta \) (QED).

3.3 Friction angle

The maximum friction angle of the forces in \( S \) is found in the vertices of \( S \).

Lemma 1: For any two forces \( c_1 \) and \( c_2 \in \mathbb{R}^3 \), let 

\[ c = (1-s)c_1 + sc_2, \quad 0 \leq s \leq 1 \] 

be a force along the line segment connecting them as shown in Fig. 4. Assume that the z-component of \( c \) is positive. Either \( c_1 \) or \( c_2 \) maximizes a friction angle along the line segment \( c \).

Proof: The proof is straightforward by focusing on the fact that a friction cone is convex. Let \( \theta_1 \) and \( \theta_2 \) be the friction angles of \( c_1 \) and \( c_2 \), respectively and let 

\[ \theta = \max(\theta_1, \theta_2) \] 

The wrench and jacobian matrices are

\[ W = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -r_1 & -r_1 & 0 & -r_2 & r_1 & -r_1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -r_2 & r_1 & 0 & -r_2 & r_1 & r_1 \end{pmatrix} \] 

(27)

\[ J^T = \begin{pmatrix} J_1^T & 0 \\ 0 & J_2^T \end{pmatrix} \] 

(28)

\[ J_1^T = \begin{pmatrix} r & -r & r_1 & -2r & r_2 & r \\ 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & -r \end{pmatrix} \] 

(29)

\[ J_2^T = \begin{pmatrix} r & r & r_1 & 2r & r_2 & -r \\ 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & -r \end{pmatrix} \] 

(30)

where \( r = 0.1/\sqrt{2} = 0.0707, \quad r_1 = 0.1 + r, \quad r_1 = 0.1 + 2r, \quad r_2 = 0.2 + r, \quad r_2 = 0.2 + 3r. \)

Given the external force/moment

\[ w_{ext} = \begin{pmatrix} 1[N] \\ 1[N] \\ 0.1[Nm] \end{pmatrix} \] 

and the joint torques([Nm])

\[ \tau_{11} = 3.368, \quad \tau_{12} = 1.949, \quad \tau_{13} = 0.624 \]

\[ \tau_{21} = -3.609, \quad \tau_{22} = -2.049, \quad \tau_{23} = -0.483, \]

the friction component of the grasp force \( f \) is

\[ f = v_t + H_c x \]

where

\[ v_t = \begin{pmatrix} -0.8473 & 2.2071 & -0.8473 \\ -1.5669 & 2.2071 & -1.5669 \end{pmatrix} \] 

(31)
and

\[ H_t = \begin{pmatrix} -0.3273 & 0.0999 & 0.3645 \\ -0.3086 & -0.0706 & -0.2578 \\ 0.3273 & -0.0999 & -0.3645 \\ 0.0000 & -0.4822 & 0.1322 \\ 0.3086 & 0.0706 & 0.2578 \\ 0.0000 & 0.4822 & -0.1322 \end{pmatrix} \]

There are eight combinations of linearly independent three rows in \( H_t \). The eight vertices are

\[
\begin{pmatrix}
  2.977 \\ -1.748 \\ 5.477 \\
  5.196 \\ -2.426 \\ 3.006
\end{pmatrix},
\begin{pmatrix}
  2.977 \\ 4.296 \\ 3.821 \\
  5.196 \\ 3.619 \\ 1.350
\end{pmatrix},
\begin{pmatrix}
  -5.196 \\ -3.619 \\ -1.350 \\
  -2.977 \\ -4.296 \\ -3.821
\end{pmatrix},
\begin{pmatrix}
  -5.196 \\ 2.426 \\ 3.006 \\
  -2.977 \\ 1.748 \\ 5.477
\end{pmatrix}.
\]

(33)

\( f_i \) is also written as \( f_i = v_i + H'_i x' \) where

\[ H'_i = H_t P^{-1} = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  -1 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & -1 & 0 \\
  0 & 0 & -1
\end{pmatrix} \]

(34)

\[ P = \begin{pmatrix}
  -0.3273 & 0.0999 & 0.3645 \\
  -0.3086 & -0.0706 & -0.2578 \\
  0.0000 & -0.4822 & 0.1322
\end{pmatrix}. \]

(35)

Since the matrix \( H'_i \) is sparse, it is not difficult to use Theorem 1 directly. Then \( f \) is NGF if

\[
\begin{align*}
(1) & \quad f_{i1} > 0, \quad f_{i3} < 0, \\
(2) & \quad f_{i1} < 0, \quad f_{i3} > 0, \\
(3) & \quad f_{i2} > 0, \quad f_{i5} < 0, \\
(4) & \quad f_{i2} < 0, \quad f_{i5} > 0, \\
(5) & \quad f_{i4} > 0, \quad f_{i6} < 0, \\
(6) & \quad f_{i4} < 0, \quad f_{i6} > 0.
\end{align*}
\]

The ranges of the elements in \( (x'_1 \ x'_2 \ x'_3)^T \) so that \( f \) is IGF are

\[
\begin{align*}
-0.8473 & \quad < x'_1 < 0.8473 \\
-2.2071 & \quad < x'_2 < 2.2071 \\
-1.5669 & \quad < x'_3 < 1.5669.
\end{align*}
\]

These define a rectangular solid with eight vertices. The vertices transformed by \( x = P^{-1} x' \) are identical to those in Eq. (33).

### 4.2 Three dimensional power grasp

Consider the four fingered power grasp in Fig. 6. Each finger has three links and placed at every 90 degrees. The distal and middle links of each finger are in contact with the grasped object (not shown). The number of contact points is \( M = 8 \) and the number of joints is \( L = 12 \).

The dimensions of the matrices \( W, J, A \) etc. are

\[
W \in R^{6 \times 24}, \quad J \in R^{24 \times 12}, \quad A \in R^{18 \times 24}
\]

\[ H \in R^{24 \times 6}, \quad T \in R^{24 \times 16}, \quad H_t \in R^{16 \times 6}. \]

There are 2688 combinations of linearly independent six rows in \( H_t \) among \( 12 \times 6 \) are identical to those in Eq. (33).

\[
\begin{align*}
&\tau_11 = \tau_{21}, \quad \tau_{31} = \tau_{41} = 0.0[Nm], \\
&\tau_{12} = \tau_{22}, \quad \tau_{32} = \tau_{42} = -0.9223[Nm], \\
&\tau_{13} = \tau_{23}, \quad \tau_{33} = \tau_{43} = -0.5270[Nm].
\end{align*}
\]

compute a friction angle for the 2688 vertices and for all the eight contact points. The maximum friction angle is 45[deg]. It takes 8.69[sec] with MATLAB M-language (not compiled) and Pentium III 600Mz.

### 5 Joint torque computation

Let \( F \) be the set of the grasp force \( f \) such that every contact force \( f_{ij} \) at the contact point \( C_{ij} \) is within a friction cone. A necessary condition that the contact points do not slip is that \( F \cap S \) exists. However this is not sufficient since there could be a force in \( F \cap S \) which is just on the boundary of the friction cone. A sufficient condition is that \( S \subset F \).

Note: A contact may be accelerated and shifts microscopically even if the contact force is strictly within a static friction cone[9]. But the traditional Coulomb static friction model assumes that the contact remains in equilibrium and it slips (macroscopically) only when the contact force reaches the static friction limit.

As shown in section 3, the maximum friction angle is found at the vertex of \( S \). Thus if all forces at the vertices of \( S \) are in \( F \), then all forces in \( S \) are in \( F \). Let \( W_p \) be a set of external wrenches to be balanced by
constant joint torques. Suppose that \( W_P \) is a convex polyhedron and let \( \mathbf{w}_{ext,l}, l = 1, \cdots, N_W \) be its vertex. From Eq. (26), the force at the \( k \)th vertex of \( S \) balancing \( \mathbf{w}_{ext,l} \) of \( W_P \) is given by

\[
\mathbf{u}_{kl} = C_k \mathbf{w}_{ext,l} + D_k \mathbf{\tau} \tag{36}
\]

Since \( W_P \) is assumed convex, if \( \mathbf{u}_{kl} \in F \) for \( l = 1, \cdots, N_W \), then for any \( \mathbf{w}_{ext} \in W_P \)

\[
\mathbf{u}_k = C_k \mathbf{w}_{ext} + D_k \mathbf{\tau} \in F. \tag{37}
\]

Therefore the problem of computing minimum constant joint torques such that \( S \in F \) for a given set of external wrenches can be formulated as follows.

\[
\min \| \mathbf{\tau} \|_2 \tag{38}
\]

such that \( \mathbf{u}_{kl} \in F \) \( \tag{39} \)

for \( k = 1, \cdots, N_S, l = 1, \cdots, N_W \)

where \( N_S \) is the number of vertices in \( S \). The constraint in (39) is linear if the fiction cone is approximated by a polyhedral cone. Thus this problem is a standard quadratic programming problem.

Example: Consider again the two fingered planar power grasp in Fig. 5. Suppose that a set of external wrenches is given by the rectangular solid in Fig. 7.

![Figure 7: A set of external wrenches](image)

When the coefficient of friction is 1.0, the minimum constant torques are given by

\[
\mathbf{\tau} = (4.575\ 2.702\ 0.936\ -4.575\ -2.702\ -0.936). \tag{40}
\]

6 Conclusion

This paper proved that the indeterminate grasp force in power grasps is within a bounded hyper-polyhedron not necessarily convex. A sufficient condition that slip does not occur considering the indeterminacy is that the forces at the vertices are within a friction cone. We presented an algorithm for computing joint torques that can balance a given set of external disturbances. An important question is whether or not slip could occur when the hyper-polyhedron of the indeterminate grasp force is not inside the friction cone but intersects with it. This topic is our future work.

References


Lemma A: If \( \operatorname{Ker} \mathbf{H}_I = \emptyset \), there exists a particular solution \( \mathbf{v}_1 \) such that \( \mathbf{T}^T \mathbf{v}_1 \) is orthogonal to any homogeneous solution in \( \operatorname{Im} \mathbf{H}_I \).

Proof: Let \( \mathbf{v}_0 \) be a particular solution. For any particular solution \( \mathbf{v}_1 \), its friction component is given by \( \mathbf{v}_{1t} = \mathbf{v}_{0t} + \mathbf{H}_i \mathbf{y} \). We show that there exists \( \mathbf{y} \) such that

\[
\mathbf{H}_i^T (\mathbf{v}_{0t} + \mathbf{H}_i \mathbf{y}) = 0 \tag{41}
\]

If \( \operatorname{Ker} \mathbf{H}_I = \emptyset \), then \( \mathbf{H}_i \) is full-column rank and the matrix \( \mathbf{H}_i^T \mathbf{H}_i \) is non-singular. Therefore we have

\[
\mathbf{y} = - (\mathbf{H}_i^T \mathbf{H}_i)^{-1} \mathbf{H}_i^T \mathbf{v}_{0t} \tag{42}
\]

(QED).

For \( \mathbf{f}_t = \mathbf{v}_{1t} \), Eq. (15) in Theorem 1 does not hold. Therefore \( \mathbf{f}_t = \mathbf{v}_{1t} \) is IGF.