AN EXACT REPRESENTATION OF EFFECTIVE CUTTING SHAPES OF 5-AXIS
CNC MACHINING USING RATIONAL BÉZIER AND B-SPLINE TOOL MOTIONS

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ABSTRACT

Presented in this paper is a new approach to 5-axis CNC tool path generation for sculptured surface machining with a flat-end cutter. Rational Bézier and B-spline motions are used to plan cutter motions so that an exact representation of the effective cutting shape can be obtained. The exact representation leads to an accurate computation of the scallop curve generated by two adjacent tool paths. Two examples are given to show how this result can be used to accurately plan and verify tool paths for 5-axis CNC milling of sculptured surfaces.

1 INTRODUCTION

It is well known that CNC machining of sculptured surfaces using flat-end cutting tools and 5-axis machines offers the benefits of higher material removal rates, better accessibility, and reduced number of set-ups. The quality of the machined surface is determined by geometric factors as well as kinematic, dynamic, and thermal properties of the machine tool. A vast amount of research has been done in the area of 5-axis tool path planning in order to achieve the competing goals of higher accuracy for manufactured surface and reduced machining time. Much of the existing work on CNC tool path generation focuses on geometric issues such as scallop heights, local and rear gouging (Vichers and Quan, 1989; Marciniak, 1991; Jensen and Anderson, 1992; Choi et al., 1993; Chen et al., 1993; Menon and Voelcker, 1993; Li and Jerard, 1994; Kim and Chu, 1994; Suresh and Yang, 1994; Lee and Chang, 1995; Lin and Koren, 1996; Sarma and Dutta, 1997; Lee, 1997 and 1998; Lee and Ji, 1997; Rao et al., 1997; Lo, 1999; Rao and Sarma, 2000).

The present paper deals also with the geometry of CNC tool path. Instead of focusing on a particular instant of the tool motion and studying local geometric issues at the instant, this paper uses the recently developed rational Bézier and B-spline motions (Ge and Ravani, 1991; Jüttler and Wagner, 1996; Srinivasan and Ge, 1998) for CNC tool path generation. The main advantages of using such freeform motions for tool path representation include (a) the entire tool path can be represented using a much more compact set of control positions of the freeform motion as opposed to a huge data set of discrete cutter positions; (b) since the tool motion representation is analytic, it may provide a framework for including kinematic and dynamic factors of the machine tool in tool path generation (Ge, 1996). Furthermore, Xia and Ge (1999) has shown that the boundary surfaces of the swept volume of a flat-end cylindrical cutter undergoing a rational Bezier or B-spline motion can be represented exactly. The present paper extends this recent result to obtain an exact representation of the effective cutting shape. This leads to an accurate computation of the scallop curve generated by two adjacent rational Bezier or B-spline tool paths. These results, when combined with existing approaches to 5-axis CNC tool path planning, would make these methods much more reliable.

The organization of the paper is as follows. The Section 2 reviews the kinematics fundamentals required for the development of the paper. Section 3 presents an analytic representation of the exact swept section of a flat-end cutting tool under a rational Bézier motion. Section 4 presents an example to show how this result can be used to plan iso-parametric rational Bézier and B-spline tool paths such that the resulting scallop heights of the entire manufactured surface do not exceed the specified scallop height. Section 5 presents another example in which near-constant scallop rational B-spline tool paths are obtained.

2 RATIONAL BÉZIER MOTIONS

This paper follows Ge and Ravani (1991) and Xia and Ge (1999) and uses dual quaternions to represent spatial motions. A dual quaternion $\mathbf{Q}$ consists of a pair of quaternions $\mathbf{Q}$ and $\mathbf{R}$, where $\mathbf{Q}$ is a quaternion of rotation and $\mathbf{R}$ is another quaternion associated with the translation component. Details on quaternions and dual quaternions can...
be found in McCarthy (1990).
In dual-quaternion representation, rigid transformations of homogeneous point coordinates, \( \tilde{P} = (\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4) \) and \( P = (P_1, P_2, P_3, P_4) \), are given by the following quaternion equations:

\[
\tilde{P} = P Q Q' + P_4 (RQ' - QR')
\]

where “\( \ast \)” denotes the conjugate of a quaternion.

Given a set of dual quaternions \( Q_i \), the following rational Bézier representation

\[
\tilde{Q}(t) = \sum_{i=0}^{n} B_i^n(t) \tilde{Q}_i
\]

defines a Bézier curve in the space of dual quaternions. The dual-quaternion curve corresponds to a rational Bézier motion whose point trajectories are rational Bézier curves.

The trajectory of a point \( P \) under a rational Bézier motion (2) can be rewritten in matrix form as

\[
\tilde{P}^{2n}(t) = [H^{2n}(t)] P
\]

where

\[
[H^{2n}(t)] = \sum_{k=0}^{2n} B_k^{2n}(t)[H_k],
\]

\[
[H_k] = \sum_{i+j=k} \frac{C_n^i C_n^j}{C_n^k} ([H_i^+] [H_j^-] + [H_j^-] [H_i^{0+}] - [H_i^+] [H_j^{0-}] ).
\]

In the above, the matrices \([H_i^+], [H_j^-], [H_i^{0+}], [H_j^{0-}] \) are given by

\[
[H_i^+] = \begin{bmatrix}
    Q_{1,4} & -Q_{1,3} & Q_{1,2} & Q_{1,1} \\
    Q_{1,3} & Q_{1,4} & -Q_{1,2} & Q_{1,1} \\
    -Q_{1,2} & Q_{1,1} & Q_{1,4} & Q_{1,3} \\
    -Q_{1,1} & -Q_{1,2} & Q_{1,3} & Q_{1,4}
\end{bmatrix},
\]

\[
[H_j^-] = \begin{bmatrix}
    Q_{j,4} & -Q_{j,3} & Q_{j,2} & -Q_{j,1} \\
    Q_{j,3} & Q_{j,4} & -Q_{j,2} & Q_{j,1} \\
    -Q_{j,2} & Q_{j,1} & Q_{j,4} & -Q_{j,3} \\
    Q_{j,1} & -Q_{j,2} & Q_{j,3} & Q_{j,4}
\end{bmatrix},
\]

\[
[H_i^{0+}] = \begin{bmatrix}
    0 & 0 & 0 & R_{i,1} \\
    0 & 0 & R_{i,1} & 0 \\
    0 & R_{i,1} & 0 & 0 \\
    0 & 0 & 0 & R_{i,4}
\end{bmatrix},
\]

\[
[H_i^{0-}] = \begin{bmatrix}
    0 & 0 & 0 & -R_{i,1} \\
    0 & 0 & -R_{i,1} & 0 \\
    0 & -R_{i,1} & 0 & 0 \\
    0 & 0 & 0 & -R_{i,4}
\end{bmatrix}.
\]

It is clear from (3) that the point trajectory is a rational Bézier curve of degree \( 2n \).

3 THE EFFECTIVE CUTTING SHAPE

The circular boundary of the base of a cylindrical cutting tool is called the cutting circle of the tool. When the tool follows a trajectory, it traces out a swept volume. Xia and Ge (1999) presented an exact representation of all boundary surfaces of a cylindrical cutter under rational Bézier and B-spline motions. Another related work is by Ji et al. and Wagner (1999).

This section focuses on the swept surface generated by a cutting circle under a rational Bézier motion. In particular, we investigate the profile of the swept surface in the cutting plane which is a plane normal to the direction of motion at a given instant. The profile is referred to as effective cutting shape or swept section in CNC machining literature. A scallop is an uncut volume left between two adjacent tool paths. The scallop height \( \delta \) is defined at the maximum of the height of the volume measured from the designed surface \( S(u,v) \). The highest point, called the scallop point, traces out a curve along the tool path, called the scallop curve. The distance between two neighboring scallop curves is called step-over distance or path interval (Sarma and Dutta, 1997). When a cutter moves along a tool path without changing its orientation, the effective cutting shape is an ellipse obtained as the intersection of the cutting plane with the cylindrical cutter. Traditionally, the ellipse has been used to represent the effective cutting shape whether or not there is change in tool orientation. However, as pointed out by Sarma (2000a), when there is change in the tool orientation, the effective cutting shape can deviate significantly from the ellipse for points that are away from the contact point (CC). Since a scallop point is away from CC and is traditionally computed as the intersection of two ellipses, the error in estimating the scallop point and the step-over distance are even greater.

We now consider the problem of obtaining an exact representation of the effective cutting shape of a cutting circle under a rational Bézier motion. First, we represent a circular arc of sweep angle 180° with a quadratic Bézier curve with one control point at infinity (see, for example, Piegl and Tiller 1995). Without loss of generality, we assume that the coordinate system is chosen such that the circle is on \( XZ \) plane with radius \( R \) and its center is the origin of the coordinate system. Then the half circle below the...
X-axis is represented as:

\[ P(s) = \sum_{i=0}^{2} B_i^2(s)P_i \]  

where \( B_i^2(s) \) are quadratic Bernstein polynomials and \( P_0 = (R, 0, 0, 1), P_1 = (0, 0, -R, 0) \) and \( P_2 = (-R, 0, 0, 1) \) are the homogeneous coordinates of the three Bézier control points. Similarly, the circular arc above \( x \)-axis can be represented by the same formula as (5), but with \( P_1 = (0, 0, R, 0) \). The arc as shown by (5) is under a rational Bézier motion as defined by (2), the swept surface of the arc can be represented in the following tensor product form:

\[ P(s, t) = [H^{2n}(t)]P(s) \]

\[ = \sum_{k=0}^{2n} \sum_{i=0}^{2} B_k^{2n}(t)B_i^2(s)[H_k]P_i \]

\[ = \sum_{k=0}^{2n} \sum_{i=0}^{2} B_k^{2n}(t)B_i^2(s)P_{ik} \]  

where \( P_{ik} = [H_k]P_i \) and the matrix \( [H_k] \) is given by (4).

Now turn our attention to a CNC tool path on a designed surface. Let \( n, t, b \) denote the normal, tangent, and binormal vectors associated with the Frenet frame at the contact point \( C \) of the tool path. The binormal plane defined by \( n \) and \( b \) captures the instantaneous position of the tool motion. Let \( M \) denote a four-dimensional vector whose coordinates are the homogeneous coordinates of the binormal plane. Then the intersection of the plane with the swept surface (6) defines the swept section at the instant \( t \). The swept section is a planar curve \( s(t) \) on the swept surface such that \( P(s(t), \cdot M = 0, i.e., \)

\[ \sum_{k=0}^{2n} \sum_{i=0}^{2} B_k^{2n}(t)B_i^2(s)(P_{ik} \cdot M) = 0. \]

This leads to

\[ \sum_{i=0}^{2} B_i^2(s)f_i = 0, \]  

where \( f_i = \sum_{k=0}^{2n} B_k^{2n}(t)(P_{ik} \cdot M) \). Solving the quadratic equation (7), we obtain

\[ s(t) = \frac{f_0 - f_1 \pm \sqrt{f_1^2 - 4f_0f_2}}{2f_1}. \]  

When \( f_1^2 - 4f_0f_2 \geq 0 \), the swept surface (6) intersects with the plane \( M \). Otherwise there is no intersection between them. The curve of intersection, \( C(t) = P(s(t), t) \), obtained by substituting (8) into (6) is the exact representation of the effective cutting shape of a flat-end cylindrical cutter under a rational Bézier motion of degree 2n.

4 PLANNING ISO-PARAMETRIC TOOL MOTIONS FOR BOUNDED SCALLOP HEIGHT

In this section we describe a procedure for planning an iso-parametric tool motions for a given sculptured surface \( S(u, v) \) such that the resulting scallop curves do not exceed the specified scallop height \( \delta \). This procedure is used as an example to show how the exact representation of the cutting shape can be used in conjunction of rational Bézier and B-spline motion for tool path planning. In the following, without loss of generality, we consider only the iso-parametric paths defined by fixing \( u = \alpha \), to obtain \( S(u, v) \).

4.1 Cutter position generation and interpolation

For a given iso-parametric curve \( S(u, v) \), we first discretize it into a set of \( n \) points \( S(u_i, v_j) \) with \( v_j(j = 0, \ldots, n-1, v_0 = 0, v_{n-1} = 1) \). We use an existing method such as the one proposed by Lee (1998) to generate a set of cutter positions on the \( S(u_i, v_j) \) and the associated tool-frames \( T_{ij} \). We then convert these tool positions from matrix representation to quaternion representation \( Q_{ij} \). After that, we use a piecewise Bézier dual quaternion curve such as a cubic B-spline dual quaternion curve \( Q(u_i, v_j) \) to interpolate the dual quaternions \( Q_{ij} \) in a manner similar to curve interpolation in CAGD (see, for example, Farin, 1997). See also Srinivasan and Ge (1998).

Once we obtained a rational Bézier or B-spline motion, we can proceed to compute the swept volume of a cylindrical cutter using the method presented in Xia and Ge (1999). For each tool motion segment \( [v_i, v_{i+1}] \), we check the deviation of the swept surface from the designed surface to determine if further subdivision within the segment \( [v_i, v_{i+1}] \) is needed. The following factors need to be examined:

- global gouging between the swept volume segment and surface segment \( [v_i, v_{i+1}] \);
- over-cut situation when the swept surface generated by the base of the cutter contributes to the boundary surfaces of the swept volume;
- If none of the above two situations arises, then check the distance from \( S(u_i, v_j) \) and the swept surface to see if it is within the specified error \( \delta \).

If any of above three cases exists, the segment \( [v_i, v_{i+1}] \)

\[ s(t) = \frac{f_0 - f_1 \pm \sqrt{f_1^2 - 4f_0f_2}}{2f_1}. \]
should be divided into smaller ones to obtain a more refined interpolating rational B-spline motion.

4.2 Estimation of step-over distance

Once we have obtained a rational B-spline motion \( Q(u_i, v) \) such that its swept surface approximates the designed surface near the iso-parametric curve \( S(u_i, v) \). The next step is to obtain the step size \( \Delta u_i \) such that the resulting scallop height generated by \( Q(u_i, v) \) and \( Q(u_{i+1}, v) \) (where \( u_{i+1} = u_i + \Delta u_i \)) is no larger than the specified scallop height \( \delta \).

The step size \( \Delta u_i \) is related to the step-over distance \( l_i \) of two adjacent tool motions by

\[
  l_i = \Delta u_i ||S_u||, \tag{9}
\]

where \( S_u = \partial S/\partial u \) and \( ||S_u|| \) denote the maximum length of the derivative vector for \( u \in [0, 1] \). For non-isoparametric motions, the step-over distance is related to step sizes \( \Delta x_i \) and \( \Delta y_j \) by (Lo, 1999):

\[
  l_{ij} = S_u \Delta u_i + S_v \Delta v_j, \tag{10}
\]

where \( S_u = \partial S/\partial u \) and \( \Delta v_j \) is the step size in \( v \) direction.

Given a specified maximum scallop height \( \delta \), the step-over distance can be estimated using a formula proposed by Lin and Koren (1996):

\[
  l_i = \sqrt{ \frac{8\delta}{\kappa_{e} + \sigma\kappa_{b}} } \tag{11}
\]

where \( \kappa_{e} \) is the effective cutting curvature of the effective cutting shape at CC, \( \kappa_{b} \) is the effective surface curvature of the designed surface \( S(u, v) \) at CC, and \( \sigma = 1 \) for convex surface and \( \sigma = -1 \) for concave surface. Since the curvatures vary when CC changes along the tool path \( S(u, v) \), the step-over distance \( l_i \) is a function of \( v \). For the example presented in this paper, we choose the value of \( l_i \) when \( v = 1/2 \) for the purpose of simplicity.

The curvature \( \kappa_{b} \) is a property of the designed surface and can be obtained using formulas presented in Faux and Pratt (1981). The effective cutting curvature has been traditionally computed using the elliptic approximation of the cutting shape, which can be significantly different from true cutting curvature. In this paper, since we have developed an exact representation of the cutting shape, we can compute the cutting curvature exactly. We note that recently Sarma (2000b) presented formulas for computing the exact cutting curvature using the incline and tilt angles of the cutting tool. The use of exact cutting curvature gives better estimation for the step-over distance.

Once the step-over distance is obtained, one can use Eq. (9) to obtain an estimate for \( \Delta u_i \). After that one can generate a neighboring rational B-spline tool motion \( Q(u_i + \Delta u_i, v) \). For a given instant \( v \), one can compute the locations of the cutting planes associated with \( Q(u_i, v) \) and \( Q(u_i + \Delta u_i, v) \). The intersection of these two planes yield a line which intersect the cutting shape in one of the planes to obtain the scallop point \( m \). As \( v \) varies in \([0, 1] \), the scallop point traces out the scallop curve \( m(u_i, v) \). The distance between the two curves \( m(u_i, v) \) and \( S(u_i, v) \) defines the scallop function \( h(v) \). If \( h(v) \leq \delta \) for all \( v \in [0, 1] \), then the estimate \( l_i \) is good. Otherwise, a new estimate needs to be generated using a search routine. In the example presented in this paper, a simple binary search is used.

5 PLANNING NEAR-CONSTANT SCALLOP RATIONAL B-SPLINE TOOL MOTIONS

In the case of iso-parametric tool path planning, we know the cutting direction because CC points move along an iso-parametric curve of \( S(u, v) \). In near constant scallop case, we do not know the next CC point, so we do not know the cutting direction. If we assume the cutting direction is along a parametric curve at each CC point, the maximum effective cutting radius can not be achieved. In the following, we propose a method which compromises between the goals of maximum cutting radius and required scallop height. Four steps are included:

1. Generate a set of cutter locations \( T_{ij} \) based on the local geometry of surface \( S(u, v) \) by using Eq. (11) and (10).
2. Obtain quaternions \( Q_{ij} \) transforming the cutter from its original location to \( T_{ij} \) and associated knot sequence \( u_{ij}, v_{ij} \). Then a two-parameter rational B-spline motion \( Q(u, v) \) is generated to interpolate all \( Q_{ij} \).
3. Fine tune the control quaternions of \( Q(u, v) \) so that the surface swept by the circular edge under \( Q(u, v) \) matches the design surface. Swept volume of cylindrical cutter under two-parameter motion \( Q(u, v) \) (see Xia and Ge, 2000) is needed to avoid global interference and make sure the swept surface generated by the base plane of the cutter does not contribute to the boundary surfaces of the swept volume, which would cause over-cut. If necessary, we raise the degree of \( Q(u, v) \) such that there are more control quaternions can be used provide additional flexibility for adjusting the two-parameter motion.
4. Assuming that \( \dot{Q}_i(t) \) is known, obtain a one-parameter motion \( Q_{i+1}(t) \) from \( Q(u, v) \) such that the scallop
height generated by cutter under $\hat{Q}_i(t)$ and $\hat{Q}_{i+1}(t)$ is within the range.

This 4th step above requires more explanation. The basic idea is as follows. First find a set of discrete points $P$ on the swept surface of the cutting circle under the existing one-parameter motion $\hat{Q}_i(t)$ such that these points maintain constant distance $\delta$ from the designed surface $S(u, v)$. Then find a set of discrete cutter locations from the two-parameter $Q(u, v)$ such that cutter bottom edges contain points $P$. Finally, interpolate all cutter positions to obtain $Q_{i+1}(t)$.

As an example, we implemented the algorithms for planning $C^2$ rational B-spline tool paths for CNC machining of the following design surface:

\[
S^x(u, v) = -2 + 6v + 2v^3,
\]
\[
S^y(u, v) = 6u,
\]
\[
S^z(u, v) = 6u + 6v - 6u^2 - 6v^2,
\]

where $u, v = [0, 1]$. Figure 1(a) shows the tool path and 1(b) is the resulting manufactured surface for iso-parametric case, Figure 2(a) shows the tool path and 2(b) is the manufactured surface for near constant scallop case.

6 CONCLUSIONS

In this paper we have developed a method for the exact representation of the effective cutting shapes for a flat-end cutter under rational Bézier or B-spline motion. We presented two examples to demonstrate how the result can be used to plan iso-parametric and near constant scallop rational tool motions.

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