Comments on “New Results on the Synthesis of PID Controllers”

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Abstract—Silva, Datta, and Bhattacharyya (in the above paper) have studied the feedback control system, provided with a proportional-integral-derivative (PID) controller and relative to a first-order plant with a time delay, and have determined the range of the PID controller parameters suitable for the stability by means of a version of the Hermite–Biehler theorem. The purpose of this note is to find the same results in a simpler way by means of the classical Nyquist’s stability criterion.

Index Terms—Closed-loop systems, delay systems, proportional-integral-derivative (PID) control, reduced-order systems, stability.

The stability zones in the coordinates system \((k_d, k_i)\) for a suitable value of \(k_p\) are quadrilaterally defined in [1] as follows.

1) Fig. 3, ([1]): \(T > 0, k_i > 0, -1 < k_p < k_p^+\)

\[
\delta_r(z_1) < 0 \quad \text{and} \quad \delta_r(z_2) > 0
\]

and

\[
\frac{k k_d}{T} < 1 \quad \text{and} \quad k_i > 0; \quad (1)
\]

2) Fig. 4, ([1]): \(T < -0.5 L, k_i < 0, k_k < k_p < -1\)

\[
\delta_r(z_1) > 0 \quad \text{and} \quad \delta_r(z_2) < 0
\]

and

\[
\frac{k k_d}{T} < 1 \quad \text{and} \quad k_i < 0; \quad (2)
\]

where \(\delta_r(z)\) and \(\delta_i(z)\), which are the real and the imaginary components of the characteristic equation multiplied by \(e^{\alpha t}\) and calculated for \(s = jz/L\), are given by

\[
\delta_r(z) = k k_p - k k_d \frac{2 z^2}{L} - \frac{z}{L} \sin(z) - T \frac{z^2}{T^2} \cos(z) \quad (3)
\]

\[
\delta_i(z) = \frac{z}{L} \left[ k k_p + \cos(z) - T \frac{z}{T} \sin(z) \right] \quad (4)
\]

and \(z_1\) and \(z_2\) are the first two positive roots of \(\delta_r(z) = 0\).

The open-loop transfer function \(F_d(s)\) and its real component \(R_{d1}(z)\), imaginary component \(I_{d1}(z)\), modulus \(M_{d1}(z)\), calculated for \(s = jz/L\), are given by

\[
F_d(s) = \frac{k e^{-\alpha s} (k_i + k_p s + k_d s^2)}{s + T s^2} \quad (8)
\]

\[
R_{d1}(z) = k (A \cos(z) + B \sin(z)) \quad (5)
\]

\[
I_{d1}(z) = k (B \cos(z) - A \sin(z)) \quad (6)
\]

\[
M_{d1}(z) = k \left( \frac{k_d^2 L^4 + (k_d^2 - 2 k_p k_d) L^2 z^2 + k_d^2 z^4}{(L^2 + T^2 z^2)^{1/2}} \right)^{1/2} \quad (7)
\]

where

\[
A = \frac{T (k_d z^2 - k_i L^2) + k_p L^2}{L^2 + T^2 z^2} \quad (8)
\]

\[
B = \frac{L k_d z^2 - k_i L^2 - k_p T z^2}{L^2 + T^2 z^2} \quad (9)
\]

Before applying Nyquist’s stability criterion, it is necessary to make the following two observations, which are the fundamental elements of this note.

1) Remark 1

Differentiating (7) with respect to \(z\), we obtain

\[
\frac{d M_{d1}(z)}{dz} = \frac{k^2 L^2 - m_0 - 2 m_2 z^2 + m_4 z^4}{M_{d1}(z)} \quad (8)
\]

where

\[
m_0 = k_i^2 L^4
\]

\[
m_2 = k_i^2 T^2 L^2
\]

\[
m_4 = k_d^2 \left( k_d^2 - 2 k_k k_d \right) T^2.
\]

Since (8) has none real root for \(m_4 < 0\) and one positive root \(z_m\) for \(m_4 > 0\) it can be stated that, when \(z\) increases from zero to infinity, \(M_{d1}(z)\) decreases always if \(m_4 < 0\) or, in contrary case, decreases for \(z\) lower than \(z_m\) and increases, for \(z\) higher than \(z_m\), up to \(M_{d1}(+\infty)\), given by

\[
M_{d1}(+\infty) = \frac{k k_d}{T} \quad (9)
\]

Therefore the point of the first part of the Nyquist diagram, lying on the negative real axis and farthest from the coordinates origin, is the first intersection or the last one \((z = +\infty)\). Moreover, \(M_{d1}(+\infty) < 1\) is one of the conditions, indicated in (1) and (2), which define the stability zones.

Remark 2

Making equal to zero the imaginary component \(I_{d1}(z)\) we obtain from (6)

\[
I_{m1}(z) = I_{m2}(z) \quad (10)
\]

where

\[
I_{m1}(z) = k k_d \frac{L^2}{z} \quad (11)
\]

\[
I_{m2}(z) = k k_d \frac{T z \cos(z) + L \sin(z)}{L \cos(z) - T \sin(z)} \quad (12)
\]

Replacing in (5) the second member of (11) with the second member of (12), the coordinate \(R_{d1}(z)\) of the points lying on the real axis can be expressed as

\[
R_{d1}(z) = \frac{k k_d L}{L \cos(z) - T \sin(z)} \quad (13)
\]

It is easy to verify that \(R_{d1}(z) = -1\) and \(\delta_i(z) = 0\) are the same comparing (13) with (4) and also that (10) may be obtained eliminating \(k\) from \(\delta_r(z) = 0\) and \(\delta_i(z) = 0\); therefore, the system of the equations \(\delta_r(z) = 0\) and \(\delta_i(z) = 0\) is equivalent to \(I_{m1}(z) = I_{m2}(z)\) and \(R_{d1}(z) = -1\) one. Moreover, the lower and the upper bounds of \(k_p\), named in [1] \(k_l (T < 0)\) and \(k_u (T > 0)\) are, respectively, the minimum and the maximum of \(k_p\), deduced from \(R_{d1}(z) = -1\), in the interval of \((0, \pi)\).

Now, we can plot the Nyquist diagrams (see Fig. 1 for \(T > 0\) and Fig. 2 for \(T < 0\) and the curves relative to (13) and to both members of (10) (see Fig. 3 for \(T > 0\) and \(k_p > 0\) and Fig. 4 for \(T < -L\)). We consider only the positive frequencies \((\omega > 0)\) in the contour \(Q\); the plot of \(F_d(s)\) for negative frequencies is obviously symmetrical to the plot for positive frequencies, with the real axis as the axis of simmetry. The first part of the Nyquist diagram, from \(z = 0\) to \(z = +\infty\), is a line, starting from the point \(R_{d1}(0) =\)
Fig. 1. Nyquist plot for $T > 0$. (a) Contour $Q$. (b) $F_a(s)$.

(a)

(b)

Fig. 2. Nyquist plot for $T < 0$. (a) Contour $Q$. (b) $F_a(s)$.

$k(k_p - (T + L)k_i) = s \ln(-Lk_i)$ and spiralling toward the circle of radius equal to $M_a(\infty) = |k_{df}/T|$.

Considering Nyquist’s stability criterion (see, for example, [2, para. 10.3.6, p. 181]) we note the following.

1) Open-loop stable plants ($T > 0$ and $k_i > 0$): Figs. 1 and 3 Since there are no poles of $F_a(s)$ in the right-half plane, for the stability it is necessary to have none rotation of $F_a(s)$ about $-1 + j0$ point, i.e., $R_0(z) > -1$ for all intersections with the negative real axis. For $k_p > 0$ since, according to Remark 1, the farthest of these intersections from the coordinates origin is the first one ($z = z_1$) or the last one ($z = +\infty$), the inequalities $z_1 < z_a < z_2$ and $M_a(\infty) < 1$ must be satisfied (see Fig. 3, where $z_1$ and $z_2$ are indicated as the coordinates of the crossing points $R_0 = -1$ and $z_0$ of the crossing point $I_m = I_{m_2}$). For $k_p < 0$ the plots of $I_{m_2}(z)$ and $R_0(z)$ are symmetrical to the shown ones in Fig. 3, with the $z$ axis as the axis of symmetry, and the inequalities $z_a < z_1$ and $z_2 < z_5$ replace the previous ones ($z_6$ is the second intersection with the negative real axis). In both cases, the stability zone is therefore defined by

$$I_{m_2}(z_1) > I_{m_2}(z_1)$$

and

$$M_a(\infty) < 1$$

and

$k_i > 0$. (14)

2) Open-loop unstable plants ($T < 0$ and $k_i < 0$): Figs. 2 and 4 Since there is one pole of $F_a(s)$ in the right-half plane, for the stability it is necessary to have one counterclockwise rotation of $F_a(s)$ about $-1 + j0$ point, i.e., $R_0(z) < -1$ for the first intersection with the negative real axis ($z = z_a$) and $R_0(z) > -1$ for all the remaining ones. Since, according to Remark 1, the farthest of these remaining intersections from the coordinates origin is the second ($z = z_6$) or the last one ($z = +\infty$), the inequalities $z_6 < z_1$, $z_6 < z_2$, and $M_a(\infty) < 1$ must be satisfied (see Fig. 4, where $z_1$ and $z_2$ are indicated as the coordinates of the crossing points $R_0 = -1$ and $z_6$ of the crossing points $I_m = I_{m_2}$). For $T < -L$ the first part of the plot of $I_{m_2}(z)$ is always decreasing, but the proof previously enunciated for $T < -L$ is fully applicable. The stability zone is therefore defined by

$$I_{m_1}(z_1) < I_{m_2}(z_1)$$

and

$$M_a(\infty) < 1$$

and

$k_i < 0$. (15)

Finally, since, according to Remark 2, (14) coincides with (1) and (15) with (2), we conclude that the results of [1] have been obtained in a simpler way.

REFERENCES
