On the Stability and Control of Nonlinear Dynamical Systems via Vector Lyapunov Functions

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***Abstract*—Vector Lyapunov theory has been developed to weaken the hypothesis of standard Lyapunov theory in order to enlarge the class of Lyapunov functions that can be used for analyzing system stability. In this paper, we extend the theory of vector Lyapunov functions by constructing a generalized comparison system whose vector field can be a function of the comparison system states as well as the nonlinear dynamical system states. Furthermore, we present a generalized convergence result which, in the case of a scalar comparison system, specializes to the classical Krasovskii–LaSalle invariant set theorem. In addition, we introduce the notion of a control vector Lyapunov function as a generalization of control Lyapunov functions, and show that asymptotic stabilizability of a nonlinear dynamical system is equivalent to the existence of a control vector Lyapunov function. Moreover, using control vector Lyapunov functions, we construct a universal decentralized feedback control law for a decentralized nonlinear dynamical system that possesses guaranteed gain and sector margins in each decentralized input channel. Furthermore, we establish connections between the recently developed notion of vector dissipativity and optimality of the proposed decentralized feedback control law. Finally, the proposed control framework is used to construct decentralized controllers for large-scale nonlinear systems with robustness guarantees against full modeling uncertainty.**

***Index Terms*—Comparison principle, control vector Lyapunov functions, decentralized control, gain and sector margins, invariance principle, inverse optimality, large-scale systems, partial stability, vector Lyapunov functions.**

I. Introduction

ONE OF THE MOST basic issues in system theory is the stability of dynamical systems. The most complete contribution to the stability analysis of nonlinear dynamical systems is due to Lyapunov [1]. Lyapunov's results, along with the Krasovskii–LaSalle invariance principle [2]–[4], provide a powerful framework for analyzing the stability of nonlinear dynamical systems. Lyapunov methods have also been used by control system designers to obtain stabilizing feedback controllers for nonlinear systems. In particular, for smooth feedback, Lyapunov-based methods were inspired by Jurdjevic and Quinn [5] who give sufficient conditions for smooth stabilization based on the ability of constructing a Lyapunov function for the closed-loop system. More recently, Artstein [6] introduced the notion of a control Lyapunov function whose existence guarantees a feedback control law which globally stabilizes a nonlinear dynamical system. In general, the feedback control law is not necessarily smooth, but can be guaranteed to be at least continuous at the origin in addition to being smooth everywhere else. Even though for certain classes of nonlinear dynamical systems a universal construction of a feedback stabilizer can be obtained using control Lyapunov functions [7], [8], there does not exist a unified procedure for finding a Lyapunov function candidate that will stabilize the closed-loop system for general nonlinear systems.

In an attempt to simplify the construction of Lyapunov functions for the analysis and control design of nonlinear dynamical systems, several researchers have resorted to vector Lyapunov functions as an alternative to scalar Lyapunov functions. Vector Lyapunov functions were first introduced by Bellman [9] and Matrosov [10], and further developed in [11]–[14], with [11]–[13] and [15]–[18] exploiting their utility for analyzing large-scale systems. The use of vector Lyapunov functions in dynamical system theory offers a very flexible framework since each component of the vector Lyapunov function can satisfy less rigid requirements as compared to a single scalar Lyapunov function. Weakening the hypothesis on the Lyapunov function enlarges the class of Lyapunov functions that can be used for analyzing system stability. In particular, each component of a vector Lyapunov function need not be positive definite with a negative or even negative–semidefinite derivative. Alternatively, the time derivative of the vector Lyapunov function need only satisfy an element-by-element inequality involving a vector field of a certain comparison system. Since in this case the stability properties of the comparison system imply the stability properties of the dynamical system, the use of vector Lyapunov theory can significantly reduce the complexity (i.e., dimensionality) of the dynamical system being analyzed. Extensions of vector Lyapunov function theory that include relaxed conditions on standard vector Lyapunov functions as well as matrix Lyapunov functions appear in [17]–[19].

In this paper, we extend the theory of vector Lyapunov functions in several directions. Specifically, we construct a generalized comparison system whose vector field can be a function of the comparison system states as well as the nonlinear dynamical system states. Next, using partial stability notions [20], [21] for the comparison system we provide sufficient conditions for stability of the nonlinear dynamical system. In addition, we present a convergence result reminiscent to the invariance principle that allows us to weaken the hypothesis on the comparison system while guaranteeing asymptotic stability of the nonlinear dynamical system via vector Lyapunov functions.

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Furthermore, we introduce the notion of a control vector Lyapunov function as a generalization of control Lyapunov functions and show that asymptotic stabilizability of a nonlinear dynamical system is equivalent to the existence of a control vector Lyapunov function. In addition, using control vector Lyapunov functions, we present a universal decentralized feedback stabilizer for a decentralized affine in the control nonlinear dynamical system with guaranteed gain and sector margins. Furthermore, we establish connections between vector dissipativity notions [22] and inverse optimality of decentralized nonlinear regulators. These results are then used to develop decentralized controllers for large-scale dynamical systems with robustness guarantees against full modeling and input uncertainty. Finally, it is important to stress that the main purpose of this paper is not to present a procedure for constructing a generalized comparison system or constructing vector Lyapunov functions, but rather to develop a new and novel analysis and control design framework for nonlinear systems based on control vector Lyapunov functions. As such, a key contribution of the paper is a control design framework for nonlinear dynamical systems predicated on existing vector Lyapunov methods.

II. Mathematical Preliminaries

In this section, we introduce notation and definitions needed for developing stability analysis and synthesis results for nonlinear dynamical systems via vector Lyapunov functions. Let $\mathbb{R}$ denote the set of real numbers, $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, $\mathbb{R}^n$ denote the set of $n\times 1$ column vectors, and $(\cdot)^{\mathrm{T}}$ denote transpose. For $v \in \mathbb{R}^q$ we write $v \geq\geq 0$ (respectively, $v \gg 0$) to indicate that every component of v is nonnegative (respectively, positive). In this case, we say that v is *nonnegative* or *positive*, respectively. Let $\overline{\mathbb{R}}_+^q$ and $\mathbb{R}_+^q$ denote the nonnegative and positive orthants of $\mathbb{R}^q$, that is, if $v \in \mathbb{R}^q$, then $v \in \overline{\mathbb{R}}_+^q$ and $v \in \mathbb{R}_+^q$ are equivalent, respectively, to $v \geq\geq 0$ and $v \gg 0$. Furthermore, let $\overset{\circ}{\mathcal{D}}$ and $\overline{\mathcal{D}}$ denote the interior and the closure of the set $\mathcal{D} \subset \mathbb{R}^n$, respectively. Finally, we write $\|\cdot\|$ for an arbitrary spatial vector norm in $\mathbb{R}^n$, $V'(x)$ for the Fréchet derivative of V at x, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball centered at α with radius ε, $\mathbf{e} \in \mathbb{R}^q$ for the ones vector, that is, $\mathbf{e} \triangleq [1,\ldots,1]^{\mathrm{T}}$, and $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$, that is, for each $\varepsilon > 0$ there exists $T > 0$ such that dist $(x(t),\mathcal{M}) < \varepsilon$ for all $t > T$, where dist $(p,\mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p-x\|$.

The following definition introduces the notion of class $\mathcal{W}$ functions involving *quasimonotone increasing* functions.

Definition 2.1 [15]: A function $w = [w_1,\ldots,w_q]^{\mathrm{T}} : \mathbb{R}^q \times \mathcal{V} \to \mathbb{R}^q$, where $\mathcal{V} \subseteq \mathbb{R}^s$, is of *class* $\mathcal{W}$ if for every fixed $y \in \mathcal{V} \subseteq \mathbb{R}^s$, $w_i(z',y) \leq w_i(z'',y)$, $i = 1,\ldots,q$, for all $z',z'' \in \mathbb{R}^q$ such that $z'_j \leq z''_j$, $z'_i = z''_i$, $j = 1,\ldots,q$, $i \neq j$, where z_i denotes the ith component of z.

If $w(\cdot,y) \in \mathcal{W}$ we say that w satisfies the *Kamke condition* [23], [24]. Note that if $w(z,y) = W(y)z$, where $W : \mathcal{V} \to \mathbb{R}^{q\times q}$, then the function $w(\cdot,y)$ is of class $\mathcal{W}$ if and only if $W(y)$ is *essentially nonnegative* for all $y \in \mathcal{V}$, that is, all the off-diagonal entries of the matrix function $W(\cdot)$ are nonnegative. Furthermore, note that it follows from Definition 2.1 that any scalar ($q = 1$) function $w(z,y)$ is of class $\mathcal{W}$.

Finally, we introduce the notion of class $\mathcal{W}_{\mathrm{d}}$ functions involving *nondecreasing* functions.

Definition 2.2 [14]: A function $w = [w_1,\ldots,w_q]^{\mathrm{T}} : \mathbb{R}^q \times \mathcal{V} \to \mathbb{R}^q$, where $\mathcal{V} \subseteq \mathbb{R}^s$, is of *class* $\mathcal{W}_{\mathrm{d}}$ if for every fixed $y \in \mathcal{V} \subseteq \mathbb{R}^s$, $w(z',y) \leq\leq w(z'',y)$ for all $z',z'' \in \mathbb{R}^q$ such that $z' \leq\leq z''$.

Note that if $w(\cdot,y) \in \mathcal{W}_{\mathrm{d}}$, then $w(\cdot,y) \in \mathcal{W}$.

III. Generalized Differential Inequalities

In this section, we develop a generalized comparison principle involving differential inequalities, wherein the underlying *comparison system* is partially dependent on the state of a dynamical system. Specifically, we consider the nonlinear comparison system given by

$$\dot{z}(t) = w(z(t),y(t)), \quad z(t_0) = z_0, \qquad t \in \mathcal{I}_{z_0} \tag{1}$$

where $z(t) \in \mathcal{Q} \subseteq \mathbb{R}^q$, $t \in \mathcal{I}_{z_0}$, is the comparison system state vector, $y : \mathcal{T} \to \mathcal{V} \subseteq \mathbb{R}^s$ is a *given* continuous function, $\mathcal{I}_{z_0} \subseteq \mathcal{T} \subseteq \overline{\mathbb{R}}_+$ is the maximal interval of existence of a solution $z(t)$ of (1), $\mathcal{Q}$ is an open set, $0 \in \mathcal{Q}$, and $w : \mathcal{Q}\times\mathcal{V} \to \mathbb{R}^q$. We assume that $w(\cdot,y(t))$ is continuous in t and satisfies the Lipschitz condition

$$\|w(z',y(t)) - w(z'',y(t))\| \leq L\|z'-z''\|, \qquad t \in \mathcal{T} \tag{2}$$

for all $z',z'' \in \mathcal{B}_\delta(z_0)$, where $\delta > 0$ and $L > 0$ is a Lipschitz constant. Hence, it follows from [25, Th. 2.2] that there exists $\tau > 0$ such that (1) has a unique solution over the time interval $[t_0, t_0+\tau]$.

Theorem 3.1: Consider the nonlinear comparison system (1). Assume that the function $w : \mathcal{Q}\times\mathcal{V} \to \mathbb{R}^q$ is continuous and $w(\cdot,y)$ is of class $\mathcal{W}$. If there exists a continuously differentiable vector function $V = [v_1,\ldots,v_q]^{\mathrm{T}} : \mathcal{I}_{z_0} \to \mathcal{Q}$ such that

$$\dot{V}(t) \ll w(V(t),y(t)), \qquad t \in \mathcal{I}_{z_0} \tag{3}$$

then $V(t_0) \ll z_0$, $z_0 \in \mathcal{Q}$, implies

$$V(t) \ll z(t), \qquad t \in \mathcal{I}_{z_0} \tag{4}$$

where $z(t)$, $t \in \mathcal{I}_{z_0}$, is the solution to (1).

Proof: Since $V(t)$, $t \in \mathcal{I}_{z_0}$, is continuous it follows that for sufficiently small $\tau > 0$:

$$V(t) \ll z(t), \qquad t \in [t_0,\, t_0+\tau]. \tag{5}$$

Now, suppose, *ad absurdum*, inequality (4) does not hold on the entire interval $\mathcal{I}_{z_0}$. Then there exists $\hat{t} \in \mathcal{I}_{z_0}$ such that $V(t) \ll z(t)$, $t \in [t_0,\hat{t})$, and for at least one $i \in \{1,\ldots,q\}$

$$v_i(\hat{t}) = z_i(\hat{t}) \tag{6}$$

and

$$v_j(\hat{t}) \leq z_j(\hat{t}), \qquad j \neq i, \quad j = 1,\ldots,q. \tag{7}$$

Since $w(\cdot,y) \in \mathcal{W}$, it follows from (3), (6), and (7) that

$$\dot{v}_i(\hat{t}) < w_i(V(\hat{t}),y(\hat{t})) \leq w_i(z(\hat{t}),y(\hat{t})) = \dot{z}_i(\hat{t}) \tag{8}$$

which, along with (6), implies that for sufficiently small $\hat{\tau} > 0$, $v_i(t) > z_i(t)$, $t \in [\hat{t} - \hat{\tau}, \hat{t})$. This contradicts the fact that $V(t) \ll z(t)$, $t \in [t_0, \hat{t})$, and establishes (4). □

Next, we present a stronger version of Theorem 3.1 where the strict inequalities are replaced by soft inequalities.

Theorem 3.2: Consider the nonlinear comparison system (1). Assume that the function $w : \mathcal{Q} \times \mathcal{V} \to \mathbb{R}^q$ is continuous and $w(\cdot, y)$ is of class $\mathcal{W}$. Let $z(t)$, $t \in \mathcal{I}_{z_0}$, be the solution to (1) and $[t_0,\, t_0 + \tau] \subseteq \mathcal{I}_{z_0}$ be a compact interval. If there exists a continuously differentiable vector function $V : [t_0,\, t_0{+}\tau] \to \mathcal{Q}$ such that

$$\dot{V}(t) \leq\leq w(V(t),\, y(t)), \qquad t \in [t_0,\, t_0 + \tau] \tag{9}$$

then $V(t_0) \leq\leq z_0$, $z_0 \in \mathcal{Q}$, implies $V(t) \leq\leq z(t)$, $t \in [t_0,\, t_0 + \tau]$.

Proof: Consider the family of comparison systems given by

$$\dot{z}(t) = w(z(t), y(t)) + \frac{\varepsilon}{n}\mathbf{e}, \quad z(t_0) = z_0 + \frac{\varepsilon}{n}\mathbf{e} \tag{10}$$

where $\varepsilon > 0, n \in \overline{\mathbb{Z}}_+$, and $t \in \mathcal{I}_{z_0+(\varepsilon/n)\mathbf{e}}$, and let the solution to (10) be denoted by $s_{(n)}(t, z_0 + (\varepsilon/n)\mathbf{e})$, $t \in \mathcal{I}_{z_0+(\varepsilon/n)\mathbf{e}}$. Now, it follows from [26, p. 17, Th. 3] that there exists a compact interval $[t_0,\, t_0 + \tau] \subseteq \mathcal{I}_{z_0}$ such that $s_{(n)}(t, z_0 + (\varepsilon/n)\mathbf{e})$, $t \in [t_0, t_0 + \tau]$, is defined for all sufficiently large n. Moreover, it follows from Theorem 3.1 that

$$V(t) \ll s_{(n)}\left(t, z_0 + \frac{\varepsilon}{n}\mathbf{e}\right) \ll s_{(m)}\left(t, z_0 + \frac{\varepsilon}{m}\mathbf{e}\right), \quad n > m \qquad t \in [t_0,\, t_0 + \tau] \tag{11}$$

for all sufficiently large $m \in \overline{\mathbb{Z}}_+$. Since the functions $s_{(n)}(t, z_0{+}(\varepsilon/n)\mathbf{e}), t \in [t_0,\, t_0{+}\tau], n \in \overline{\mathbb{Z}}_+$, are continuous in t, decreasing in n, and bounded from below, it follows that the sequence of functions $s_{(n)}(\cdot, z_0{+}(\varepsilon/n)\mathbf{e})$ converges uniformly on the compact interval $[t_0,\, t_0 + \tau]$ as $n \to \infty$, that is, there exists a continuous function $\hat{z} : [t_0,\, t_0 + \tau] \to \mathcal{Q}$ such that

$$s_{(n)}\left(t, z_0 + \frac{\varepsilon}{n}\mathbf{e}\right) \to \hat{z}(t), \qquad n \to \infty \tag{12}$$

uniformly on $[t_0,\, t_0 + \tau]$. Hence, it follows from (11) and (12) that

$$V(t) \leq\leq \hat{z}(t), \qquad t \in [t_0,\, t_0 + \tau]. \tag{13}$$

Next, note that it follows from (10) that

$$s_{(n)}\left(t, z_0 + \frac{\varepsilon}{n}\mathbf{e}\right) = z_0 + \frac{\varepsilon}{n}\mathbf{e} + \int_{t_0}^{t} w\left(s_{(n)}\left(\sigma, z_0 + \frac{\varepsilon}{n}\mathbf{e}\right), y(\sigma)\right) \mathrm{d}\sigma \tag{14}$$

for all $t \in [t_0,\, t_0 {+} \tau]$, which implies that $\hat{z}(t_0) = z_0$ and, since $y(\cdot)$ and $w(\cdot, \cdot)$ are continuous, $w(s_{(n)}(t, z_0{+}(\varepsilon/n)\mathbf{e}), y(t)) \to w(\hat{z}(t), y(t))$ as $n \to \infty$ uniformly on $[t_0,\, t_0{+}\tau]$. Hence, taking the limit as $n \to \infty$ on both sides of (14) yields

$$\hat{z}(t) = z_0 + \int_{t_0}^{t} w(\hat{z}(\sigma), y(\sigma))\mathrm{d}\sigma, \qquad t \in [t_0,\, t_0 + \tau] \tag{15}$$

which implies that $\hat{z}(t)$ is the solution to (1) on the interval $[t_0,\, t_0 + \tau]$. Hence, by uniqueness of solutions of (1) we obtain that $\hat{z}(t) = z(t)$, $[t_0,\, t_0 + \tau]$. This along with (13) proves the result. □

Next, consider the nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0, \qquad t \in \mathcal{I}_{x_0} \tag{16}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \in \mathcal{I}_{x_0}$, is the system state vector, $\mathcal{I}_{x_0}$ is the maximal interval of existence of a solution $x(t)$ of (16), $\mathcal{D}$ is an open set, $0 \in \mathcal{D}$, and $f(\cdot)$ is Lipschitz continuous on $\mathcal{D}$. The following result is a direct consequence of Theorem 3.2.

Corollary 3.1: Consider the nonlinear dynamical system (16). Assume there exists a continuously differentiable vector function $V : \mathcal{D} \to \mathcal{Q} \subseteq \mathbb{R}^q$ such that

$$V'(x)f(x) \leq\leq w(V(x), x), \qquad x \in \mathcal{D} \tag{17}$$

where $w : \mathcal{Q} \times \mathcal{D} \to \mathbb{R}^q$ is a continuous function, $w(\cdot, x) \in \mathcal{W}$, and

$$\dot{z}(t) = w(z(t), x(t)), \quad z(t_0) = z_0, \qquad t \in \mathcal{I}_{z_0,x_0} \tag{18}$$

has a unique solution $z(t)$, $t \in \mathcal{I}_{z_0,\,x_0}$, where $x(t)$, $t \in \mathcal{I}_{x_0}$, is a solution to (16). If $[t_0,\, t_0 + \tau] \subseteq \mathcal{I}_{x_0} \cap \mathcal{I}_{z_0,\,x_0}$ is a compact interval, then $V(x_0) \leq\leq z_0$, $z_0 \in \mathcal{Q}$, implies $V(x(t)) \leq\leq z(t)$, $t \in [t_0,\, t_0 + \tau]$.

Proof: For any given $x_0 \in \mathcal{D}$, the solution $x(t)$, $t \in \mathcal{I}_{x_0}$, to (16) is a well defined function of time. Hence, define $\eta(t) \triangleq V(x(t))$, $t \in \mathcal{I}_{x_0}$, and note that (17) implies

$$\dot{\eta}(t) \leq\leq w(\eta(t), x(t)), \qquad t \in \mathcal{I}_{x_0}. \tag{19}$$

Moreover, if $[t_0,\, t_0 + \tau] \subseteq \mathcal{I}_{x_0} \cap \mathcal{I}_{z_0,\,x_0}$ is a compact interval, then it follows from Theorem 3.2, with $y(t) \equiv x(t)$ and $V(x_0) = \eta(t_0) \leq\leq z_0$, that $V(x(t)) = \eta(t) \leq\leq z(t)$, $t \in [t_0,\, t_0 + \tau]$, which establishes the result. □

If in (16), $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz continuous, then (16) has a unique solution $x(t)$ for all $t \geq t_0$. A more restrictive sufficient condition for global existence and uniqueness of solutions to (16) is continuous differentiability of $f : \mathbb{R}^n \to \mathbb{R}^n$ and uniform boundedness of $f'(x)$ on $\mathbb{R}^n$. Note that if the solutions to (16) and (18) are globally defined for all $x_0 \in \mathcal{D}$ and $z_0 \in \mathcal{Q}$, then the result of Corollary 3.1 holds for any arbitrarily large but compact interval $[t_0,\, t_0 + \tau] \subset \overline{\mathbb{R}}_+$. For the remainder of this paper, we assume that the solutions to the systems (16) and (18) are defined for all $t \geq t_0$. Continuous differentiability of $f(\cdot)$ and $w(\cdot, \cdot)$ provides a sufficient condition for the existence and uniqueness of solutions to (16) and (18) for all $t \geq t_0$.

IV. Stability Theory via Vector Lyapunov Functions

In this section, we develop a generalized vector Lyapunov function framework for the stability analysis of nonlinear dynamical systems using the generalized comparison principle developed in Section III. Specifically, consider the cascade nonlinear dynamical system given by

$$\dot{z}(t) = w(z(t), x(t)), \quad z(t_0) = z_0, \qquad t \geq t_0 \tag{20}$$

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0 \tag{21}$$

where $z_0 \in \mathcal{Q} \subseteq \mathbb{R}^q$, $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$, $[z^{\mathrm{T}}(t), x^{\mathrm{T}}(t)]^{\mathrm{T}}$, $t \geq t_0$, is the solution to (20) and (21), $w : \mathcal{Q} \times \mathcal{D} \to \mathbb{R}^q$ is continuous, $w(\cdot, x) \in \mathcal{W}$, $w(0,0) = 0$, $f : \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous on $\mathcal{D}$, and $f(0) = 0$. The following definition involving the notion of partial stability is needed for the next result.

Definition 4.1 [21]: The nonlinear dynamical system $\mathcal{G}$ given by (20) and (21) is *Lyapunov stable with respect to* z if, for every $\varepsilon > 0$ and $x_0 \in \mathcal{D}$, there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that $\|z_0\| < \delta$ implies that $\|z(t)\| < \varepsilon$ for all $t \geq t_0$. $\mathcal{G}$ is *Lyapunov stable with respect to* z *uniformly in* x_0 if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|z_0\| < \delta$ implies that $\|z(t)\| < \varepsilon$ for all $t \geq t_0$ and for all $x_0 \in \mathcal{D}$. $\mathcal{G}$ is *asymptotically stable with respect to* z if it is Lyapunov stable with respect to z and, for every $x_0 \in \mathcal{D}$, there exists $\delta = \delta(x_0) > 0$ such that $\|z_0\| < \delta$ implies that $\lim_{t\to\infty} z(t) = 0$. $\mathcal{G}$ is *asymptotically stable with respect to* z *uniformly in* x_0 if it is Lyapunov stable with respect to z uniformly in x_0 and there exists $\delta > 0$ such that $\|z_0\| < \delta$ implies that $\lim_{t\to\infty} z(t) = 0$ for all $x_0 \in \mathcal{D}$. $\mathcal{G}$ is *exponentially stable with respect to* z *uniformly in* x_0 if there exist positive scalars α, β, and δ such that $\|z_0\| < \delta$ implies that $\|z(t)\| \leq \alpha \|z_0\| e^{-\beta(t-t_0)}$, $t \geq t_0$, for all $x_0 \in \mathcal{D}$. Finally, $\mathcal{G}$ is *globally asymptotically* (respectively, *exponentially*) *stable with respect to* z *uniformly in* x_0 if the previous two statements hold for all $z_0 \in \mathbb{R}^q$ and $x_0 \in \mathbb{R}^n$.

Theorem 4.1: Consider the nonlinear dynamical system (16). Assume that there exist a continuously differentiable vector function $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$ and a positive vector $p \in \mathbb{R}_+^q$ such that $V(0) = 0$, the scalar function $v : \mathcal{D} \to \overline{\mathbb{R}}_+$ defined by $v(x) \triangleq p^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, is such that $v(x) > 0$, $x \neq 0$, and

$$V'(x) f(x) \leq\leq w(V(x), x), \qquad x \in \mathcal{D} \tag{22}$$

where $w : \mathcal{Q} \times \mathcal{D} \to \mathbb{R}^q$ is continuous, $w(\cdot, x) \in \mathcal{W}$, and $w(0,0) = 0$. Then, the following statements hold.

- i) If the nonlinear dynamical system (20), (21) is Lyapunov stable with respect to z uniformly in x_0, then the zero solution $x(t) \equiv 0$ to (16) is Lyapunov stable.
- ii) If the nonlinear dynamical system (20), (21) is asymptotically stable with respect to z uniformly in x_0, then the zero solution $x(t) \equiv 0$ to (16) is asymptotically stable.
- iii) If $\mathcal{D} = \mathbb{R}^n$, $\mathcal{Q} = \mathbb{R}^q$, $v : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ is radially unbounded, and the nonlinear dynamical system (20), (21) is globally asymptotically stable with respect to z uniformly in x_0, then the zero solution $x(t) \equiv 0$ to (16) is globally asymptotically stable.
- iv) If there exist constants $\nu \geq 1$, $\alpha > 0$, and $\beta > 0$ such that $v : \mathcal{D} \to \overline{\mathbb{R}}_+$ satisfies

$$\alpha \|x\|^\nu \leq v(x) \leq \beta \|x\|^\nu, \qquad x \in \mathcal{D} \tag{23}$$

  and the nonlinear dynamical system (20), (21) is exponentially stable with respect to z uniformly in x_0, then the zero solution $x(t) \equiv 0$ to (16) is exponentially stable.
- v) If $\mathcal{D} = \mathbb{R}^n$, $\mathcal{Q} = \mathbb{R}^q$, there exist constants $\nu \geq 1$, $\alpha > 0$, and $\beta > 0$ such that $v : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ satisfies (23), and the nonlinear dynamical system (20), (21) is globally exponentially stable with respect to z uniformly in x_0, then the zero solution $x(t) \equiv 0$ to (16) is globally exponentially stable.

Proof: Assume there exist a continuously differentiable vector function $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$ and a positive vector $p \in \mathbb{R}_+^q$ such that $v(x) = p^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, is positive definite, that is, $v(0) = 0$ and $v(x) > 0$, $x \neq 0$. Note that since $v(x) = p^{\mathrm{T}} V(x) \leq \max_{i=1,\ldots,q}\{p_i\} \mathbf{e}^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, the function $\mathbf{e}^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, is also positive definite. Thus, there exist $r > 0$ and class $\mathcal{K}$ functions [27] α, $\beta : [0, r] \to \overline{\mathbb{R}}_+$ such that $\mathcal{B}_r(0) \subset \mathcal{D}$ and

$$\alpha(\|x\|) \leq \mathbf{e}^{\mathrm{T}} V(x) \leq \beta(\|x\|), \qquad x \in \mathcal{B}_r(0). \tag{24}$$

- i) Let $\varepsilon > 0$ and choose $0 < \hat{\varepsilon} < \min\{\varepsilon, r\}$. It follows from Lyapunov stability of the nonlinear dynamical system (20), (21) with respect to z uniformly in x_0 that there exists $\mu = \mu(\hat{\varepsilon}) = \mu(\varepsilon) > 0$ such that if $\|z_0\|_1 < \mu$, where $\|\cdot\|_1$ denotes the absolute sum norm, then $\|z(t)\|_1 < \alpha(\hat{\varepsilon})$, $t \geq t_0$, for any $x_0 \in \mathcal{D}$. Now, choose $z_0 = V(x_0) \geq\geq 0$, $x_0 \in \mathcal{D}$. Since $V(x)$, $x \in \mathcal{D}$, is continuous, the function $\mathbf{e}^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, is also continuous. Hence, for $\mu = \mu(\hat{\varepsilon}) > 0$ there exists $\delta = \delta(\mu(\hat{\varepsilon})) = \delta(\varepsilon) > 0$ such that $\delta < \hat{\varepsilon}$, and if $\|x_0\| < \delta$, then $\mathbf{e}^{\mathrm{T}} V(x_0) = \mathbf{e}^{\mathrm{T}} z_0 = \|z_0\|_1 < \mu$, which implies that $\|z(t)\|_1 < \alpha(\hat{\varepsilon})$, $t \geq t_0$. Now, with $z_0 = V(x_0) \geq\geq 0$, $x_0 \in \mathcal{D}$, and the assumption that $w(\cdot, x) \in \mathcal{W}$, $x \in \mathcal{D}$, it follows from (22) and Corollary 3.1 that $0 \leq\leq V(x(t)) \leq\leq z(t)$ on any compact interval $[t_0, t_0 + \tau]$, and hence, $\mathbf{e}^{\mathrm{T}} z(t) = \|z(t)\|_1$, $t \in [t_0, t_0 + \tau]$. Let $\tau > t_0$ be such that $x(t) \in \mathcal{B}_r(0)$, $t \in [t_0, t_0 + \tau]$, for all $x_0 \in \mathcal{B}_\delta(0)$. Thus, using (24), if $\|x_0\| < \delta$, then

$$\alpha(\|x(t)\|) \leq \mathbf{e}^{\mathrm{T}} V(x(t)) \leq \mathbf{e}^{\mathrm{T}} z(t) < \alpha(\hat{\varepsilon}), \;\; t \in [t_0, t_0 + \tau] \tag{25}$$

  which implies $\|x(t)\| < \hat{\varepsilon} < \varepsilon$, $t \in [t_0, t_0 + \tau]$. Now, suppose, *ad absurdum*, that for some $x_0 \in \mathcal{B}_\delta(0)$ there exists $\hat{t} > t_0 + \tau$ such that $\|x(\hat{t})\| = \hat{\varepsilon}$. Then, for $z_0 = V(x_0)$ and the compact interval $[t_0, \hat{t}]$ it follows from (22) and Corollary 3.1 that $V(x(\hat{t})) \leq\leq z(\hat{t})$, which implies that $\alpha(\hat{\varepsilon}) = \alpha(\|x(\hat{t})\|) \leq \mathbf{e}^{\mathrm{T}} V(x(\hat{t})) \leq \mathbf{e}^{\mathrm{T}} z(\hat{t}) < \alpha(\hat{\varepsilon})$. This is a contradiction, and hence, for a given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x_0 \in \mathcal{B}_\delta(0)$, $\|x(t)\| < \varepsilon$, $t \geq t_0$, which implies Lyapunov stability of the zero solution $x(t) \equiv 0$ to (16).

ii) It follows from *i*) and the asymptotic stability of the nonlinear dynamical system (20), (21) with respect to z uniformly in x_0 that the zero solution to (16) is Lyapunov stable and there exists $\mu > 0$ such that if $\|z_0\|_1 < \mu$, then $\lim_{t\to\infty} z(t) = 0$ for any $x_0 \in \mathcal{D}$. As in *i*), choose $z_0 = V(x_0) \geq\geq 0$, $x_0 \in \mathcal{D}$. It follows from Lyapunov stability of the zero solution to (16) and the continuity of $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$ that there exists $\delta = \delta(\mu) > 0$ such that if $\|x_0\| < \delta$, then $\|x(t)\| < r$, $t \geq t_0$, and $\mathbf{e}^{\mathrm{T}} V(x_0) = \mathbf{e}^{\mathrm{T}} z_0 = \|z_0\|_1 < \mu$. Thus, by asymptotic stability of (20), (21) with respect to z uniformly in x_0, for any arbitrary $\varepsilon > 0$ there exists $T = T(\varepsilon) > t_0$ such that $\|z(t)\|_1 < \alpha(\varepsilon)$, $t \geq T$. Thus, it follows from (22) and Corollary 3.1 that $0 \leq\leq V(x(t)) \leq\leq z(t)$ on any compact interval $[t_0, T+\tau]$, and hence, $\mathbf{e}^{\mathrm{T}} z(t) = \|z(t)\|_1$, $t \in [t_0, T+\tau]$, and, by (24)

$$\alpha(\|x(t)\|) \leq \mathbf{e}^{\mathrm{T}} V(x(t)) \leq \mathbf{e}^{\mathrm{T}} z(t) < \alpha(\varepsilon), \quad t \in [T, T+\tau]. \tag{26}$$

Now, suppose, *ad absurdum*, that for some $x_0 \in \mathcal{B}_\delta(0)$, $\lim_{t\to\infty} x(t) \neq 0$, that is, there exists a sequence $\{t_k\}_{k=1}^{\infty}$, with $t_k \to \infty$ as $k \to \infty$, such that $\|x(t_k)\| \geq \hat{\varepsilon}$, $k \in \overline{\mathbb{Z}}_+$, for some $0 < \hat{\varepsilon} < r$. Choose $\varepsilon = \hat{\varepsilon}$ and the interval $[T, T+\tau]$ such that at least one $t_k \in [T, T+\tau]$. Then it follows from (26) that $\alpha(\varepsilon) \leq \alpha(\|x(t_k)\|) < \alpha(\varepsilon)$, which is a contradiction. Hence, there exists $\delta > 0$ such that for all $x_0 \in \mathcal{B}_\delta(0)$, $\lim_{t\to\infty} x(t) = 0$ which, along with Lyapunov stability, implies asymptotic stability of the zero solution $x(t) \equiv 0$ to (16).

iii) Suppose $\mathcal{D} = \mathbb{R}^n$, $\mathcal{Q} = \mathbb{R}^q$, $v : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ is a radially unbounded function, and the nonlinear dynamical system (20), (21) is globally asymptotically stable with respect to z uniformly in x_0. In this case, for $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$ the inequality (24) holds for all $x \in \mathbb{R}^n$, where the functions $\alpha, \beta : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ are of class $\mathcal{K}_\infty$ [27]. Furthermore, Lyapunov stability of the zero solution $x(t) \equiv 0$ to (16) follows from *i*). Next, for any $x_0 \in \mathbb{R}^n$ and $z_0 = V(x_0) \in \overline{\mathbb{R}}_+^q$, identical arguments as in *ii*) can be used to show that $\lim_{t\to\infty} x(t) = 0$, which proves global asymptotic stability of the zero solution $x(t) \equiv 0$ to (16).

iv) Suppose (23) holds. Since $p \in \mathbb{R}_+^q$, then

$$\hat{\alpha}\|x\|^\nu \leq \mathbf{e}^{\mathrm{T}} V(x) \leq \hat{\beta}\|x\|^\nu, \qquad x \in \mathcal{D} \tag{27}$$

where $\hat{\alpha} \triangleq \alpha / \max_{i=1,\ldots,q}\{p_i\}$ and $\hat{\beta} \triangleq \beta / \min_{i=1,\ldots,q}\{p_i\}$. It follows from the exponential stability of the nonlinear dynamical system (20), (21) with respect to z uniformly in x_0 that there exist positive constants γ, μ, and η such that if $\|z_0\|_1 < \mu$, then

$$\|z(t)\|_1 \leq \gamma\|z_0\|_1 e^{-\eta(t-t_0)}, \qquad t \geq t_0 \tag{28}$$

for all $x_0 \in \mathcal{D}$. Choose $z_0 = V(x_0) \geq\geq 0$, $x_0 \in \mathcal{D}$. By continuity of $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$, there exists $\delta = \delta(\mu) > 0$ such that for all $x_0 \in \mathcal{B}_\delta(0)$, $\mathbf{e}^{\mathrm{T}} V(x_0) = \mathbf{e}^{\mathrm{T}} z_0 = \|z_0\|_1 < \mu$. Furthermore, it follows from (22), (27), (28), and Corollary 3.1 that, for all $x_0 \in \mathcal{B}_\delta(0)$, the inequality

$$\begin{aligned}\hat{\alpha}\|x(t)\|^\nu \leq \mathbf{e}^{\mathrm{T}} V(x(t)) \leq \mathbf{e}^{\mathrm{T}} z(t) &\leq \gamma\|z_0\|_1 e^{-\eta(t-t_0)} \\ &\leq \gamma\hat{\beta}\|x_0\|^\nu e^{-\eta(t-t_0)}\end{aligned} \tag{29}$$

holds on any compact interval $[t_0, t_0+\tau]$. This in turn implies that, for any $x_0 \in \mathcal{B}_\delta(0)$

$$\|x(t)\| \leq \left(\frac{\gamma\hat{\beta}}{\hat{\alpha}}\right)^{1/\nu} \|x_0\| e^{-(\eta/\nu)(t-t_0)}, \qquad t \in [t_0, t_0+\tau]. \tag{30}$$

Now, suppose, *ad absurdum*, that for some $x_0 \in \mathcal{B}_\delta(0)$ there exists $\hat{t} > t_0 + \tau$ such that $\|x(\hat{t})\| > \left(\gamma\hat{\beta}/\hat{\alpha}\right)^{1/\nu} \|x_0\| e^{-(\eta/\nu)(\hat{t}-t_0)}$. Then for the compact interval $[t_0, \hat{t}]$, it follows from (30) that $\|x(\hat{t})\| \leq \left(\gamma\hat{\beta}/\hat{\alpha}\right)^{1/\nu} \|x_0\| e^{-(\eta/\nu)}(\hat{t} - t_0)$, which is a contradiction. Thus, inequality (30) holds for all $t \geq t_0$ establishing exponential stability of the zero solution $x(t) \equiv 0$ to (16).

v) The proof is identical to the proof of iv).

□

If $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$ satisfies the conditions of Theorem 4.1 we say that $V(x)$, $x \in \mathcal{D}$, is a *vector Lyapunov function* [15]. Note that for stability analysis each component of a vector Lyapunov function need not be positive definite with a negative definite or negative-semidefinite time derivative along the trajectories of (20), (21). This provides more flexibility in searching for a vector Lyapunov function as compared to a scalar Lyapunov function for addressing the stability of nonlinear dynamical systems. It is important to note here that comparison systems with vector fields dependent on the states of both the system dynamics and the comparison system have been addressed in the literature [12], [28], [29], with [12] providing stability analysis using partial stability notions. However, a key difference between our formulation and the results given in [12] is in the definitions of partial stability used to analyze the stability of the generalized comparison system. Specifically, the partial stability definitions used in [12] (see Definitions 2 and 3 on pages 161 and 162) require that the entire initial system state of the generalized comparison system lie in a neighborhood of the origin, whereas in our definition of partial stability the initial system state corresponding to (21) can be arbitrary. This weaker assumption leads to stronger results. Furthermore, in Theorem 4.1 each component of the vector Lyapunov function is dependent on the entire state x of the dynamical system, while in Theorem 29 of [12] (see p. 210) the vector Lyapunov function is component-decoupled, that is, $V(x) = [v_1(x_1), \ldots, v_q(x_q)]^{\mathrm{T}}$, $x_i \in \mathbb{R}^{n_i}$, $i = 1, \ldots, q$. In addition, in Theorem 4.1 we only require that the scalar function $v(x) \triangleq p^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, be positive definite, while in [12, Th. 29] each component of a vector Lyapunov function $V(x)$, $x \in \mathcal{D}$, is assumed to be a positive–definite function of its argument.

Remark 4.1: Sufficient conditions for partial stability of the nonlinear dynamical system (20), (21) are given in [21]. Specifically, [21, Th. 1] establishes partial stability of (20), (21) in terms of a scalar Lyapunov function that is dependent on both the states z and x. Alternatively, [21, Cor. 1] provides partial stability of (20), (21) in terms of a scalar Lyapunov function that is only dependent on the comparison system state z which can simplify the stability analysis. In this case, the expanded dimension of the system (20), (21) does not introduce additional complexity for the partial stability analysis of the generalized comparison system. As in standard vector Lyapunov theory, this ensures a reduced dimension for the analysis of the comparison system while addressing a more general class of nonlinear systems. This point is further illustrated in Section VII.

The following corollary to Theorem 4.1 is immediate and corresponds to the standard vector Lyapunov theorem addressed in the literature [15].

Corollary 4.1: Consider the nonlinear dynamical system (16). Assume that there exist a continuously differentiable vector function $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}^q_+$ and a positive vector $p \in \mathbb{R}^q_+$ such that $V(0) = 0$, the scalar function $v : \mathcal{D} \to \overline{\mathbb{R}}_+$ defined by $v(x) \triangleq p^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, is such that $v(x) > 0$, $x \neq 0$, and

$$V'(x)f(x) \leq\leq w(V(x)), \qquad x \in \mathcal{D} \tag{31}$$

where $w : \mathcal{Q} \to \mathbb{R}^q$ is continuous, $w(\cdot) \in \mathcal{W}$, and $w(0) = 0$. Then, the stability properties of the zero solution $z(t) \equiv 0$ to

$$\dot{z}(t) = w(z(t)), \quad z(t_0) = z_0, \qquad t \geq t_0 \tag{32}$$

where $z_0 \in \mathcal{Q}$, imply the corresponding stability properties of the zero solution $x(t) \equiv 0$ to (16).

Proof: The proof is a direct consequence of Theorem 4.1 with $w(z, x) \equiv w(z)$. □

Next, we present a convergence result via vector Lyapunov functions that allows us to establish asymptotic stability of the nonlinear dynamical system (16) using weaker conditions than those assumed in Theorem 4.1.

Theorem 4.2: Consider the nonlinear dynamical system (16), assume that there exist a continuously differentiable vector function $V = [v_1, \ldots, v_q]^{\mathrm{T}} : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}^q_+$ and a positive vector $p \in \mathbb{R}^q_+$ such that $V(0) = 0$, the scalar function $v : \mathcal{D} \to \overline{\mathbb{R}}_+$ defined by $v(x) \triangleq p^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, is such that $v(x) > 0$, $x \neq 0$, and

$$V'(x)f(x) \leq\leq w(V(x), x), \qquad x \in \mathcal{D} \tag{33}$$

where $w : \mathcal{Q} \times \mathcal{D} \to \mathbb{R}^q$ is continuous, $w(\cdot, x) \in \mathcal{W}_{\mathrm{d}}$, and $w(0, 0) = 0$, such that the nonlinear dynamical system (20), (21) is Lyapunov stable with respect to z uniformly in x_0. Let $\mathcal{R}_i \triangleq \{x \in \mathcal{D} : v_i'(x)f(x) - w_i(V(x), x) = 0\}$, $i = 1, \ldots, q$. Then there exists $\mathcal{D}_{\mathrm{c}} \subset \mathcal{D}$ such that $x(t) \to \mathcal{R} \triangleq \cap_{i=1}^q \mathcal{R}_i$ as $t \to \infty$ for all $x(t_0) = x_0 \in \mathcal{D}_{\mathrm{c}}$. Moreover, if $\mathcal{R}$ contains no trajectory other than the trivial trajectory, then the zero solution $x(t) \equiv 0$ to (16) is asymptotically stable.

Proof: Since the nonlinear dynamical system (20), (21) is Lyapunov stable with respect to z uniformly in x_0, it follows that there exists $\hat{\delta} > 0$ such that if $\|z_0\|_1 < \hat{\delta}$, then the partial system trajectories $z(t)$, $t \geq t_0$, of (20), (21) are bounded for all $x_0 \in \mathcal{D}$. Furthermore, since $V(x)$, $x \in \mathcal{D}$, is continuous, it follows that there exists $\delta_1 = \delta_1(\hat{\delta}) > 0$ such that $\mathbf{e}^{\mathrm{T}} V(x_0) < \hat{\delta}$ for all $x_0 \in \mathcal{B}_{\delta_1}(0)$. In addition, it follows from Theorem 4.1 that the zero solution $x(t) \equiv 0$ to (16) is Lyapunov stable, and hence, for a given $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(0) \subset \overset{\circ}{\mathcal{D}}$ there exists $\delta_2 = \delta_2(\varepsilon) > 0$ such that if $x_0 \in \mathcal{B}_{\delta_2}(0)$, then $x(t) \in \mathcal{B}_\varepsilon(0)$, $t \geq t_0$, where $x(t)$, $t \geq t_0$, is the solution to (16). Choose $\delta = \min\{\delta_1, \delta_2\}$ and define $\mathcal{D}_{\mathrm{c}} \triangleq \mathcal{B}_\delta(0) \subset \mathcal{D}$. Then for all $z_0 = V(x_0)$ and $x_0 \in \mathcal{D}_{\mathrm{c}}$, it follows that $x(t) \in \mathcal{B}_\varepsilon(0)$, $t \geq t_0$, and $z(t)$, $t \geq t_0$, is bounded.

Next, consider the function

$$W_i(x, t) \triangleq v_i(x) - \int_{t_0}^{t} w_i(V(x(s)), x(s)) \mathrm{d}s, \qquad t \geq t_0$$
$$x \in \mathcal{D}, \qquad i = 1, \ldots, q. \tag{34}$$

It follows from (33) that

$$\dot{W}_i(x(t), t) = v_i'(x(t))f(x(t)) - w_i(V(x(t)), x(t)) \leq 0$$
$$t \geq t_0, \qquad x_0 \in \mathcal{D} \tag{35}$$

which implies that $W_i(x(t), t)$, $i \in \{1, \ldots, q\}$, is a nonincreasing function of time, and hence, $\lim_{t\to\infty} W_i(x(t), t)$, $i \in \{1, \ldots, q\}$, exists. Moreover, $W_i(x(t_0), t_0) = v_i(x(t_0)) < +\infty$ for all $x(t_0) = x_0 \in \mathcal{D}$ since $v_i(x)$, $x \in \mathcal{D}$, $i \in \{1, \ldots, q\}$, is continuous. Now suppose, *ad absurdum*, that for some initial condition $x(t_0) = x_0 \in \mathcal{D}_{\mathrm{c}}$, $\lim_{t\to\infty} W_i(x(t), t) = -\infty$, $i \in \{1, \ldots, q\}$. Since the function $v_i(x)$, $x \in \mathcal{D}$, $i \in \{1, \ldots, q\}$, is continuous on the compact set $\overline{\mathcal{B}_\varepsilon(0)}$, it follows that $v_i(x(t))$, $t \geq t_0$, is bounded and, hence, $\lim_{t\to\infty} \int_{t_0}^{t} w_i(V(x(s)), x(s)) \mathrm{d}s = +\infty$, $i \in \{1, \ldots, q\}$. Now, it follows from (33) and Corollary 3.1 that $V(x(t)) \leq\leq z(t)$, $t \geq t_0$, for $z(t_0) = V(x(t_0))$. Note that since $x_0 \in \mathcal{D}_{\mathrm{c}}$ it follows that $z(t)$, $t \geq t_0$, is bounded. Furthermore, since $w(\cdot, x) \in \mathcal{W}_{\mathrm{d}}$ it follows that

$$\begin{aligned} v_i(x(t)) &\leq v_i(x(t_0)) + \int_{t_0}^{t} w_i(V(x(s)), x(s)) \mathrm{d}s \\ &\leq z_i(t_0) + \int_{t_0}^{t} w_i(z(s), x(s)) \mathrm{d}s \\ &= z_i(t) \end{aligned} \tag{36}$$

for all $t \geq t_0$. Since $z(t)$, $t \geq t_0$, is bounded and $v_i(x)$, $x \in \mathcal{D}, i \in \{1, \ldots, q\}$, is continuous, it follows that there exists $M > 0$ such that $\left|\int_{t_0}^{t} w_i(V(x(s)), x(s)) \mathrm{d}s\right| < M < +\infty$, $t \geq t_0$, $i \in \{1, \ldots, q\}$. This is a contradiction and, hence, $\lim_{t\to\infty} W_i(x(t), t)$, $i \in \{1, \ldots, q\}$, exists and is finite for every $x_0 \in \mathcal{D}_{\mathrm{c}}$. Thus, for every $x_0 \in \mathcal{D}_{\mathrm{c}}$ and $t \geq t_0$, it follows that

$$\begin{aligned} &\int_{t_0}^{t} \dot{W}_i(x(s), s) \mathrm{d}s \\ &= \int_{t_0}^{t} v_i'(x(s))f(x(s)) \mathrm{d}s - \int_{t_0}^{t} w_i(V(x(s)), x(s)) \mathrm{d}s \\ &= W_i(x(t), t) - W_i(x_0, t_0) \end{aligned} \tag{37}$$

and, hence, for all $i \in \{1, \ldots, q\}$,

$$\lim_{t\to\infty} \int_{t_0}^{t} [v_i'(x(s))f(x(s)) - w_i(V(x(s)), x(s))] \mathrm{d}s \tag{38}$$

exists and is finite.

Next, since $f(\cdot)$ is Lipschitz continuous on $\mathcal{D}$ and $x(t) \in \mathcal{B}_\varepsilon(0)$ for all $x_0 \in \mathcal{D}_c$ and $t \geq t_0$ it follows that

$$\begin{aligned}\|x(t_2) - x(t_1)\| &= \left\| \int_{t_1}^{t_2} f(x(s)) \mathrm{d}s \right\| \\ &\leq L \int_{t_1}^{t_2} \|x(s)\| \,\mathrm{d}s \\ &\leq L\varepsilon(t_2 - t_1), \qquad t_2 \geq t_1 \geq t_0 \quad (39)\end{aligned}$$

where L is the Lipschitz constant on $\mathcal{D}_c$. Thus, it follows from (39) that for any $\gamma > 0$ there exists $\mu = \mu(\gamma) = \gamma/L\varepsilon$ such that $\|x(t_2) - x(t_1)\| < \gamma$, $|t_2 - t_1| < \mu$, which shows that $x(t)$, $t \geq t_0$, is uniformly continuous. Next, since $x(t)$ is uniformly continuous and $v_i'(x)f(x) - w_i(V(x), x)$, $x \in \mathcal{D}$, $i \in \{1, \ldots, q\}$, is continuous, it follows that $v_i'(x(t))f(x(t)) - w_i(V(x(t)), x(t))$, $i \in \{1, \ldots, q\}$, is uniformly continuous at every $t \geq t_0$. Hence, it follows from Barbalat's Lemma [25, p. 192] that $v_i'(x(t))f(x(t)) - w_i(V(x(t)), x(t)) \to 0$ as $t \to \infty$ for all $x_0 \in \mathcal{D}_c$ and $i \in \{1, \ldots, q\}$. Repeating the previous analysis for all $i = 1, \ldots, q$, it follows that $x(t) \to \mathcal{R} = \cap_{i=1}^q \mathcal{R}_i$ for all $x_0 \in \mathcal{D}_c$. Finally, if $\mathcal{R}$ contains no trajectory other than the trivial trajectory, then $\mathcal{R} = \{0\}$ and, hence, $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathcal{D}_c$, which proves asymptotic stability of the zero solution $x(t) \equiv 0$ to (16). □

Remark 4.2: Note that $\mathcal{R} = \cap_{i=1}^q \mathcal{R}_i \neq \varnothing$ since $0 \in \mathcal{R}$. Furthermore, recall that for every bounded solution $x(t)$, $t \geq t_0$, to (16) with initial condition $x(t_0) = x_0$, the positive limit set $\omega(x_0)$ of (16) is a nonempty, compact, invariant, and connected set with $x(t) \to \omega(x_0)$ as $t \to \infty$. If $q = 1$ and $w(V(x), x) \equiv 0$, then it can be shown that the Lyapunov derivative $\dot{V}(x)$ vanishes on the positive limit set $\omega(x_0)$, $x_0 \in \mathcal{D}_c$, so that $\omega(x_0) \in \mathcal{R}$. Moreover, since $\omega(x_0)$ is a positively invariant set with respect to (16), it follows that for all $x_0 \in \mathcal{D}_c$, the trajectory of (16) converges to the largest invariant set $\mathcal{M}$ contained in $\mathcal{R}$. In this case, Theorem 4.2 specializes to the classical Krasovskii–LaSalle invariant set theorem [4].

Remark 4.3: If for some $k \in \{1, \ldots, q\}$, $w_k(V(x), x) \equiv 0$ and $v_k'(x)f(x) < 0$, $x \in \mathcal{D}$, $x \neq 0$, then $\mathcal{R} = \mathcal{R}_k = \{0\}$. In this case, it follows from Theorem 4.2 that the zero solution $x(t) \equiv 0$ to (16) is asymptotically stable. Note that even though for $k \in \{1, \ldots, q\}$ the time derivative $\dot{v}_k(x)$, $x \in \mathcal{D}$, is negative definite, the function $v_k(x)$, $x \in \mathcal{D}$, can be nonnegative definite, in contrast to classical Lyapunov stability theory, to ensure asymptotic stability of (16).

Finally, we give a generalization of the converse Lyapunov theorem that establishes the existence of a vector Lyapunov function for an asymptotically stable nonlinear dynamical system. This result is used in the next section to establish the equivalence between asymptotic stabilizability and the existence of a control vector Lyapunov function.

Theorem 4.3: Consider the nonlinear dynamical system (16). Let $\delta > 0$ and $\mathcal{D}_0 = \mathcal{B}_\delta(0) \subset \mathcal{D}$, and assume that $f : \mathcal{D} \to \mathbb{R}^n$ is continuously differentiable and the zero solution $x(t) \equiv 0$ to (16) is asymptotically stable. Then there exist a continuously differentiable componentwise positive definite vector function $V = [v_1, \ldots, v_q]^{\mathrm{T}} : \mathcal{D}_0 \to \overline{\mathbb{R}}_+^q$ and a continuous function $w = [w_1, \ldots, w_q]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ such that $V(0) = 0, w(\cdot) \in \mathcal{W}$, $w(0) = 0$, $V'(x)f(x) \leq\leq w(V(x))$, $x \in \mathcal{D}_0$, and the zero solution $z(t) \equiv 0$ to

$$\dot{z}(t) = w(z(t)), \quad z(t_0) = z_0, \qquad t \geq t_0 \tag{40}$$

where $z_0 \in \overline{\mathbb{R}}_+^q$, is asymptotically stable.

Proof: Since the zero solution $x(t) \equiv 0$ to (16) is asymptotically stable it follows from [25, Th. 3.14] that there exist a continuously differentiable positive definite function $\tilde{v} : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ and class $\mathcal{K}$ functions [27] $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ such that

$$\alpha(\|x\|) \leq \tilde{v}(x) \leq \beta(\|x\|), \qquad x \in \mathcal{D}_0 \tag{41}$$

$$\tilde{v}'(x)f(x) \leq -\gamma(\|x\|), \qquad x \in \mathcal{D}_0. \tag{42}$$

Furthermore, it follows from (41) and (42) that

$$\tilde{v}'(x)f(x) \leq -\gamma \circ \beta^{-1}(\tilde{v}(x)), \qquad x \in \mathcal{D}_0 \tag{43}$$

where "$\circ$" denotes the composition operator and $\beta^{-1} : [0, \beta(\delta)] \to \overline{\mathbb{R}}_+$ is the inverse function of $\beta(\cdot)$, and hence, $\beta^{-1}(\cdot)$ and $\gamma \circ \beta^{-1}(\cdot)$ are class $\mathcal{K}$ functions. Next, define $V = [v_1, \ldots, v_q]^{\mathrm{T}} : \mathcal{D}_0 \to \overline{\mathbb{R}}_+^q$ such that $v_i(x) \triangleq \tilde{v}(x)$, $x \in \mathcal{D}_0$, $i = 1, \ldots, q$. Then, it follows that $V(0) = 0$ and $V'(x)f(x) \leq\leq w(V(x))$, $x \in \mathcal{D}_0$, where $w = [w_1, \ldots, w_q]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ is such that $w_i(V(x)) = -\gamma \circ \beta^{-1}(v_i(x))$, $x \in \mathcal{D}_0$. Note that $w(\cdot) \in \mathcal{W}$ and $w(0) = 0$. To show that the zero solution $z(t) \equiv 0$ to (40) is asymptotically stable, consider the Lyapunov function candidate $\hat{v}(z) \triangleq \mathbf{e}^{\mathrm{T}} z$, $z \in \overline{\mathbb{R}}_+^q$. Note that $\hat{v}(0) = 0$, $\hat{v}(z) > 0$, $z \in \overline{\mathbb{R}}_+^q$, $z \neq 0$, and $\dot{\hat{v}}(z) = -\sum_{i=1}^q \gamma \circ \beta^{-1}(z_i) < 0$, $z \in \overline{\mathbb{R}}_+^q$, $z \neq 0$. Thus, the zero solution $z(t) \equiv 0$ to (40) is asymptotically stable which completes the proof. □

V. Control Vector Lyapunov Functions

In this section, we consider a feedback control problem and introduce the notion of a control vector Lyapunov function as a generalization of control Lyapunov functions. Specifically, consider the nonlinear controlled dynamical system given by

$$\dot{x}(t) = F(x(t), u(t)), \quad x(t_0) = x_0, \qquad t \geq t_0 \tag{44}$$

where $x_0 \in \mathcal{D}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$, $t \geq t_0$, is the control input, and $F : \mathcal{D} \times \mathcal{U} \to \mathbb{R}^n$ is Lipschitz continuous for all $(x, u) \in \mathcal{D} \times \mathcal{U}$ and satisfies $F(0, 0) = 0$. We assume that the control input $u(\cdot)$ in (44) is restricted to the class of *admissible controls* consisting of measurable functions $u(\cdot)$ such that $u(t) \in \mathcal{U}$ for all $t \geq t_0$, where the constraint set $\mathcal{U}$ is given with $0 \in \mathcal{U}$. Furthermore, we assume that $u(\cdot)$ satisfies sufficient regularity conditions such that the nonlinear dynamical system (44) has a unique solution forward in time. A measurable mapping $\phi : \mathcal{D} \to \mathcal{U}$ satisfying $\phi(0) = 0$ is called a *control law*. Furthermore, if $u(t) = \phi(x(t))$, where ϕ is a control law and $x(t)$, $t \geq t_0$, satisfies (44), then $u(\cdot)$ is called a *feedback control law*.

Definition 5.1: If there exist a continuously differentiable vector function $V = [v_1, \ldots, v_q]^{\mathrm{T}} : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$, a continuous function $w = [w_1, \ldots, w_q]^{\mathrm{T}} : \mathcal{Q} \times \mathcal{D} \to \mathbb{R}^q$, and a positive vector $p \in \mathbb{R}_+^q$ such that $V(0) = 0$,

$v(x) \triangleq p^{\mathrm{T}}V(x)$, $x \in \mathcal{D}$, is positive definite, $w(\cdot, x) \in \mathcal{W}$, $w(0,0) = 0$, $\mathcal{F}(x) \triangleq \cap_{i=1}^{q} \mathcal{F}_i(x) \neq \emptyset$, $x \in \mathcal{D}$, $x \neq 0$, where $\mathcal{F}_i(x) \triangleq \{u \in \mathcal{U} : v_i'(x)F(x,u) < w_i(V(x),x)\}$, $x \in \mathcal{D}$, $x \neq 0$, $i = 1, \ldots, q$, then the vector function $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$ is called a *control vector Lyapunov function candidate*.

It follows from Definition 5.1 that if there exists a control vector Lyapunov function candidate, then there exists a feedback control law $\phi : \mathcal{D} \to \mathcal{U}$ such that

$$V'(x)F(x,\phi(x)) \ll w(V(x),x), \qquad x \in \mathcal{D}, \quad x \neq 0. \quad (45)$$

Moreover, if the nonlinear dynamical system

$$\dot{z}(t) = w(z(t), x(t)), \quad z(t_0) = z_0, \qquad t \geq t_0 \quad (46)$$
$$\dot{x}(t) = F(x(t), \phi(x(t))), \quad x(t_0) = x_0 \quad (47)$$

where $z_0 \in \mathcal{Q}$ and $x_0 \in \mathcal{D}$, is asymptotically stable with respect to z uniformly in x_0, then it follows from Theorem 4.1 that the zero solution $x(t) \equiv 0$ to (47) is asymptotically stable. In this case, the vector function $V : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ given in Definition 5.1 is called a *control vector Lyapunov function*. Furthermore, if $\mathcal{D} = \mathbb{R}^n$, $\mathcal{Q} = \mathbb{R}^q$, $\mathcal{U} = \mathbb{R}^m$, $v : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ is radially unbounded, and the system (46), (47) is globally asymptotically stable with respect to z uniformly in x_0, then the zero solution $x(t) \equiv 0$ to (44) is globally asymptotically stabilizable.

Remark 5.1: If in Definition 5.1 $w(z,x) = w(z)$ and the zero solution $z(t) \equiv 0$ to

$$\dot{z}(t) = w(z(t)), \quad z(t_0) = z_0, \qquad t \geq t_0 \quad (48)$$

where $z_0 \in \mathcal{Q}$, is asymptotically stable, then it follows from Corollary 4.1 that $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$ is a control vector Lyapunov function.

Conversely, suppose that there exists a stabilizing feedback control law $\phi : \mathcal{D} \to \mathcal{U}$ such that the zero solution $x(t) \equiv 0$ to (47) is asymptotically stable. Then it follows from Theorem 4.3 that there exist a continuously differentiable vector function $V = [v_1, \ldots, v_q]^{\mathrm{T}} : \mathcal{D}_0 \to \overline{\mathbb{R}}_+^q$, a continuous function $w = [w_1, \ldots, w_q]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, and a positive vector $p \in \mathbb{R}_+^q$ such that $V(0) = 0$, the scalar function $v : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ defined by $v(x) \triangleq p^{\mathrm{T}}V(x)$, $x \in \mathcal{D}_0$, is positive definite, $w(\cdot) \in \mathcal{W}$, $w(0) = 0$, and $V'(x)F(x,\phi(x)) \ll w(V(x))$, $x \in \mathcal{D}_0$, $x \neq 0$. Thus, $\mathcal{F}(x) \neq \emptyset$, $x \in \mathcal{D}_0$, $x \neq 0$. Moreover, since, by Theorem 4.3, the zero solution $z(t) \equiv 0$ to (48) is asymptotically stable, it follows from Remark 5.1 that $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+^q$ is a control vector Lyapunov function. Hence, a given nonlinear dynamical system of the form (44) is feedback asymptotically stabilizable if and only if there exists a control vector Lyapunov function.

In the case where $q = 1$ and $w(z,x) \equiv w(z)$, Definition 5.1 implies the existence of a positive-definite continuously differentiable function $v : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+$ and a continuous function $w : \mathcal{Q} \to \mathbb{R}$, where $\mathcal{Q} \subseteq \mathbb{R}$, such that $w(0) = 0$ and $\mathcal{F}(x) = \{u \in \mathcal{U} : v'(x)F(x,u) < w(v(x))\} \neq \emptyset$, $x \in \mathcal{D}$, $x \neq 0$, which is equivalent to

$$\inf_{u \in \mathcal{U}} v'(x)F(x,u) < w(v(x)), \qquad x \in \mathcal{D}, \quad x \neq 0. \quad (49)$$

Now, (49) implies the existence of a feedback control law $\phi : \mathcal{D} \to \mathcal{U}$ such that $v'(x)F(x,\phi(x)) < w(v(x))$, $x \in \mathcal{D}$, $x \neq 0$. Moreover, if $v : \mathcal{D} \to \overline{\mathbb{R}}_+$ is a control vector Lyapunov function (with $q = 1$), then it follows from Remark 5.1 that the zero solution $z(t) \equiv 0$ to the system (48) is asymptotically stable and, since $q = 1$, this implies that $w(z) < 0$, $z \in \mathcal{Q} \cap \overline{\mathbb{R}}_+$, $z \neq 0$. Thus, since $v(\cdot)$ is positive definite, (49) can be rewritten as

$$\inf_{u \in \mathcal{U}} v'(x)F(x,u) < 0, \qquad x \in \mathcal{D}, \quad x \neq 0 \quad (50)$$

which is equivalent to the standard definition of a control Lyapunov function [6].

Next, consider the case where the control input to (44) possesses a decentralized control architecture so that the dynamics of (44) are given by

$$\dot{x}_i(t) = F_i(x(t), u_i(t)), \qquad t \geq t_0, \quad i = 1, \ldots, q \quad (51)$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $x(t) = [x_1^{\mathrm{T}}(t), \ldots, x_q^{\mathrm{T}}(t)]^{\mathrm{T}}$, $u_i(t) \in \mathcal{U}_i \subseteq \mathbb{R}^{m_i}$, $t \geq t_0$, $\sum_{i=1}^{q} n_i = n$, and $\sum_{i=1}^{q} m_i = m$. Note that $x_i(t) \in \mathbb{R}^{n_i}$, $t \geq t_0$, $i = 1, \ldots, q$, as long as $x(t) \in \mathcal{D}$, $t \geq t_0$, and the set of control inputs is given by $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_q \subseteq \mathbb{R}^m$. In the case of a component decoupled control vector Lyapunov function candidate, that is, $V(x) = [v_1(x_1), \ldots, v_q(x_q)]^{\mathrm{T}}$, $x \in \mathcal{D}$, it suffices to require in Definition 5.1 that

$$\mathcal{F}_i(x) = \{u \in \mathcal{U} : v_i'(x_i)F_i(x,u_i) < w_i(V(x),x)\} \neq \emptyset$$
$$x \in \mathcal{D}, \qquad x \neq 0, \quad i = 1, \ldots, q \quad (52)$$

to ensure that $\mathcal{F}(x) = \cap_{i=1}^{q} \mathcal{F}_i(x) \neq \emptyset$, $x \in \mathcal{D}$, $x \neq 0$. Note that for a component decoupled control vector Lyapunov function $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$, (52) holds if and only if

$$\inf_{u \in \mathcal{U}} V'(x)F(x,u) \ll w(V(x),x), \qquad x \in \mathcal{D}, \quad x \neq 0 \quad (53)$$

where the infimum in (53) is taken componentwise, that is, for each component of (53) the infimum is calculated separately. It follows from (53) that there exists a feedback control law $\phi : \mathcal{D} \to \mathcal{U}$ such that $\phi(x) = [\phi_1^{\mathrm{T}}(x), \ldots, \phi_q^{\mathrm{T}}(x)]^{\mathrm{T}}$, $x \in \mathcal{D}$, where $\phi_i : \mathcal{D} \to \mathcal{U}_i$, and $v_i'(x_i)F_i(x,\phi_i(x)) < w_i(V(x),x)$, $x \in \mathcal{D}$, $x \neq 0$, $i = 1, \ldots, q$.

Remark 5.2: If $w_i(V(x),x) = 0$ for $x \in \mathcal{D}$ with $x_i = 0$, then condition (52) holds for all $x \in \mathcal{D}$ such that $x_i \neq 0$.

Next, we consider the special case of a nonlinear dynamical system of the form (51) with affine control inputs given by

$$\dot{x}_i(t) = f_i(x(t)) + G_i(x(t))u_i(t), \quad t \geq t_0, \quad i = 1, \ldots, q \quad (54)$$

where $f_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ satisfying $f_i(0) = 0$ and $G_i : \mathbb{R}^n \to \mathbb{R}^{n_i \times m_i}$ are smooth functions (at least continuously differentiable mappings) for all $i = 1, \ldots, q$, and $u_i(t) \in \mathbb{R}^{m_i}$, $t \geq t_0$, $i = 1, \ldots, q$.

Theorem 5.1: Consider the controlled nonlinear dynamical system given by (54). If there exist a continuously differentiable, component decoupled vector function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, a continuous function $w = [w_1, \ldots, w_q]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \times \mathbb{R}^n \to \mathbb{R}^q$, and a positive vector $p \in \mathbb{R}_+^q$ such that $V(0) = 0$, the scalar function $v : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ defined by $v(x) \triangleq p^{\mathrm{T}}V(x)$, $x \in \mathbb{R}^n$, is positive definite and radially unbounded, $w(\cdot,x)\in\mathcal{W}$, $w(0,0)=0$, and

$$v_i'(x_i)f_i(x) < w_i(V(x),x), \qquad x\in\mathcal{R}_i, \quad i=1,\ldots,q \tag{55}$$

where $\mathcal{R}_i \triangleq \{x\in\mathbb{R}^n, x\neq 0 : v_i'(x_i)G_i(x)=0\}$, $i=1,\ldots,q$, then $V:\mathbb{R}^n\to\overline{\mathbb{R}}_+^q$ is a control vector Lyapunov function candidate. If, in addition, there exists $\phi:\mathbb{R}^n\to\mathcal{U}$ such that $\phi(x)=[\phi_1^{\rm T}(x),\ldots,\phi_q^{\rm T}(x)]^{\rm T}$, $x\in\mathbb{R}^n$, and the system (46), (47) is globally asymptotically stable with respect to z uniformly in x_0, then the zero solution $x(t)\equiv 0$ to (47) is globally asymptotically stable and $V:\mathbb{R}^n\to\overline{\mathbb{R}}_+^q$ is a control vector Lyapunov function.

Proof: Note that for all $i=1,\ldots,q$,

$$\begin{aligned}\inf_{u_i\in\mathbb{R}^{m_i}} & v_i'(x_i)(f_i(x)+G_i(x)u_i)\\ &= \begin{cases}-\infty, & x\notin\mathcal{R}_i\\ v_i'(x_i)f_i(x), & x\in\mathcal{R}_i\end{cases}\\ &< w_i(V(x),x), \qquad x\in\mathbb{R}^n\end{aligned} \tag{56}$$

which implies (53). Now, the proof is a direct consequence of the definition of a control vector Lyapunov function by noting the equivalence between (52) and (53) for component decoupled vector Lyapunov functions. □

Using Theorem 5.1 we can construct an explicit feedback control law that is a function of the control vector Lyapunov function $V(\cdot)$. Specifically, consider the feedback control law $\phi(x)=[\phi_1^{\rm T}(x),\ldots,\phi_q^{\rm T}(x)]^{\rm T}$, $x\in\mathbb{R}^n$, given by (57), as shown at the bottom of the page, where $\alpha_i(x)\triangleq v_i'(x_i)f_i(x)$, $x\in\mathbb{R}^n$, $\beta_i(x)\triangleq G_i^{\rm T}(x)v_i'^{\rm T}(x_i)$, $x\in\mathbb{R}^n$, and $c_{0i}>0$, $i=1,\ldots,q$. The derivative $\dot{V}(\cdot)$ along the trajectories of the dynamical system (54), with $u=\phi(x)$, $x\in\mathbb{R}^n$, given by (57), is given by (58), as shown at the bottom of the page.

Thus, if the zero solution $z(t)\equiv 0$ to (46), (47) is globally asymptotically stable with respect to z uniformly in x_0, then it follows from Theorem 4.1 that the zero solution $x(t)\equiv 0$ to (54) with $u=\phi(x)=[\phi_1^{\rm T}(x),\ldots,\phi_q^{\rm T}(x)]^{\rm T}$, $x\in\mathbb{R}^n$, given by (57) is globally asymptotically stable.

Remark 5.3: If in Theorem 5.1 $w(z,x)=w(z)$ and the zero solution $z(t)\equiv 0$ to (48) is globally asymptotically stable, then it follows from Corollary 4.1 that the feedback control law given by (57) is a globally asymptotically stabilizing controller for the nonlinear dynamical system (54).

Remark 5.4: In the case where $q=1$, the function $w(\cdot,\cdot)$ in Theorem 5.1 can be set to be identically zero, that is, $w(z,x)\equiv 0$. In this case, the feedback control law (57) specializes to Sontag's universal formula [7] and is a global stabilizer for (54).

Since $f_i(\cdot)$ and $G_i(\cdot)$ are smooth and $v_i(\cdot)$ is continuously differentiable for all $i=1,\ldots,q$, it follows that $\alpha_i(x)$ and $\beta_i(x)$, $x\in\mathbb{R}^n$, $i=1,\ldots,q$, are continuous functions, and hence, $\phi_i(x)$ given by (57) is continuous for all $x\in\mathbb{R}^n$ if either $\beta_i(x)\neq 0$ or $\alpha_i(x)-w_i(V(x),x)<0$ for all $i=1,\ldots,q$. Hence, the feedback control law given by (57) is continuous everywhere except for the origin. The following result provides necessary and sufficient conditions under which the feedback control law given by (57) is guaranteed to be continuous at the origin in addition to being continuous everywhere else.

Proposition 5.1: The feedback control law $\phi(x)$ given by (57) is continuous on $\mathbb{R}^n$ if and only if for every $\varepsilon>0$, there exists $\delta>0$ such that for all $0<\|x\|<\delta$ there exists $u_i\in\mathbb{R}^{m_i}$ such that $\|u_i\|<\varepsilon$ and $\alpha_i(x)+\beta_i^{\rm T}(x)u_i<w_i(V(x),x)$, $i=1,\ldots,q$.

Proof: First note that since $v_i(x_i)$, $x_i\in\mathbb{R}^{n_i}$, is a nonnegative function and $v_i(0)=0$, it follows from a Taylor series expansion about $x_i=0$ that $v_i'(0)=0$, $i=1,\ldots,q$, and hence, $\phi(0)=0$. To show necessity assume that the feedback control law given by (57) is continuous on $\mathbb{R}^n$, that is, $\phi_i(x)$ is continuous on $\mathbb{R}^n$ for all $i=1,\ldots,q$. Then for every $\varepsilon>0$, there exists $\delta>0$ such that $\|\phi_i(x)\|<\varepsilon$ for all $0<\|x\|<\delta$ and, by (57), $\alpha_i(x)+\beta_i^{\rm T}(x)\phi_i(x)<w_i(V(x),x)$, $i=1,\ldots,q$. Thus, necessity follows with $u_i=\phi_i(x)$, $i=1,\ldots,q$.

To show sufficiency, assume that for $\varepsilon>0$, there exists $\delta>0$ such that for all $0<\|x\|<\delta$ there exists $u_i\in\mathbb{R}^{m_i}$ such that $\|u_i\|<\varepsilon$ and $\alpha_i(x)+\beta_i(x)u_i<w_i(V(x),x)$, $i=1,\ldots,q$. In this case, since $\|u_i\|<\varepsilon$ it follows from the Cauchy-Schwartz inequality that $\alpha_i(x)-w_i(V(x),x)<\varepsilon\|\beta_i(x)\|$, $i=1,\ldots,q$. Furthermore, since $v_i(\cdot)$, $i=1,\ldots,q$, is continuously differentiable and $G_i(\cdot)$, $i=1,\ldots,q$, is continuous it follows that there exists $\hat{\delta}>0$ such that for all $0<\|x\|<\hat{\delta}$, $\|\beta_i(x)\|<\varepsilon$, $i=1,\ldots,q$. Hence, for all $0<\|x\|<\delta_{\min}$, where $\delta_{\min}\triangleq\min\{\delta,\hat{\delta}\}$, it follows that $\alpha_i(x)-w_i(V(x),x)<\varepsilon\|\beta_i(x)\|$ and $\|\beta_i(x)\|<\varepsilon$, $i=1,\ldots,q$. Furthermore, if $\beta_i(x)=0$, then $\|\phi_i(x)\|=0$, and if $\beta_i(x)\neq 0$, then it follows from (57) that

$$\begin{aligned}\|\phi_i(x)\| &\le c_{0i}\|\beta_i(x)\| + \frac{(\alpha_i(x)-w_i(V(x),x))}{\|\beta_i(x)\|}\\ &\quad + \frac{\sqrt{(\alpha_i(x)-w_i(V(x),x))^2+(\beta_i^{\rm T}(x)\beta_i(x))^2}}{\|\beta_i(x)\|}\\ &\le \frac{2(\alpha_i(x)-w_i(V(x),x))+(c_{0i}+1)\|\beta_i(x)\|^2}{\|\beta_i(x)\|}\\ &\le (c_{0i}+3)\varepsilon, \qquad 0<\|x\|<\delta_{\min}\\ &\quad \alpha_i(x)>w_i(V(x),x), \qquad i=1,\ldots,q\end{aligned} \tag{59}$$

$$\phi_i(x) = \begin{cases}-\left(c_{0i}+\frac{(\alpha_i(x)-w_i(V(x),x))+\sqrt{(\alpha_i(x)-w_i(V(x),x))^2+(\beta_i^{\rm T}(x)\beta_i(x))^2}}{\beta_i^{\rm T}(x)\beta_i(x)}\right)\beta_i(x), & \beta_i(x)\neq 0\\ 0, & \beta_i(x)=0\end{cases} \tag{57}$$

$$\begin{aligned}\dot{v}_i(x_i) &= v_i'(x_i)(f_i(x)+G_i(x)\phi_i(x)) = \alpha_i(x)+\beta_i^{\rm T}(x)\phi_i(x)\\ &= \begin{cases}-c_{0i}\beta_i^{\rm T}(x)\beta_i(x)-\sqrt{(\alpha_i(x)-w_i(V(x),x))^2+(\beta_i^{\rm T}(x)\beta_i(x))^2}+w_i(V(x),x), & \beta_i(x)\neq 0\\ \alpha_i(x), & \beta_i(x)=0\end{cases}\\ &< w_i(V(x),x), \qquad x\in\mathbb{R}^n\end{aligned} \tag{58}$$

and

$$\begin{aligned}\|\phi_i(x)\| &\le c_{0i}\|\beta_i(x)\| + \frac{(\alpha_i(x)-w_i(V(x),x))}{\|\beta_i(x)\|}\\ &\quad + \frac{\sqrt{(\alpha_i(x)-w_i(V(x),x))^2+(\beta_i^{\mathrm{T}}(x)\beta_i(x))^2}}{\|\beta_i(x)\|}\\ &\le c_{0i}\|\beta_i(x)\| + \frac{\beta_i^{\mathrm{T}}(x)\beta_i(x)}{\|\beta_i(x)\|}\\ &= (c_{0i}+1)\|\beta_i(x)\|\\ &\le (c_{0i}+1)\varepsilon, \qquad 0<\|x\|<\delta_{\min}\\ &\quad \alpha_i(x)\le w_i(V(x),x), \qquad i=1,\ldots,q. \end{aligned} \tag{60}$$

Hence, it follows that for every $\hat{\varepsilon} \triangleq (c_{0i}+3)\varepsilon > 0$ there exists $\delta_{\min} > 0$ such that, for all $\|x\| < \delta_{\min}$, $\|\phi_i(x)\| < \hat{\varepsilon}$, which implies that $\phi_i(\cdot)$, $i=1,\ldots,q$, is continuous at the origin, and hence, $\phi(\cdot) = [\phi_1^{\mathrm{T}}(\cdot),\ldots,\phi_q^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ is continuous at the origin. □

VI. Stability Margins, Inverse Optimality, and Vector Dissipativity

In this section, we show that the feedback control law given by (57) is robust to sector bounded input nonlinearities. Specifically, we consider the nonlinear dynamical system (54) with nonlinear uncertainties in the input so that the dynamics of the system are given by

$$\dot{x}_i(t) = f_i(x(t)) + G_i(x(t))\sigma_i(u_i(t)), \;\; t\ge t_0, \;\; i=1,\ldots,q \tag{61}$$

where

$$\sigma_i(\cdot)\in\Phi_i \triangleq \Big\{\sigma_i:\mathbb{R}^{m_i}\to\mathbb{R}^{m_i} : \sigma_i(0)=0 \text{ and } \frac{1}{2}u_i^{\mathrm{T}}u_i \le \sigma_i^{\mathrm{T}}(u_i)u_i<\infty,\, u_i\in\mathbb{R}^{m_i}\Big\}$$

$i=1,\ldots,q$. In addition, we show that for the dynamical system (54) the feedback control law given by (57) is inverse optimal in the sense that it minimizes a derived performance functional over the set of stabilizing controllers $\mathcal{S}(x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x(t)\to 0 \text{ as } t\to\infty\}$.

Theorem 6.1: Consider the nonlinear dynamical system (61) and assume that the conditions of Theorem 5.1 hold with $w(z,x)\equiv w(z)$, and with the zero solution $z(t)\equiv 0$ to (48) being globally asymptotically stable. Then with the feedback control law given by (57) the nonlinear dynamical system (61) is globally asymptotically stable for all $\sigma_i(\cdot)\in\Phi_i$, $i=1,\ldots,q$. Moreover, for the dynamical system (54) the feedback control law (57) minimizes the performance functional given by

$$J(x_0,u(\cdot)) = \int_{t_0}^{\infty}\sum_{i=1}^{q}[L_{1i}(x(t)) + u_i^{\mathrm{T}}(t)R_{2i}(x(t))u_i(t)]\mathrm{d}t \tag{62}$$

in the sense that

$$J(x_0,\phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0,u(\cdot)), \qquad x_0\in\mathbb{R}^n \tag{63}$$

where $L_{1i}(x) \triangleq -\alpha_i(x) + (\gamma_i(x)/2)\beta_i^{\mathrm{T}}(x)\beta_i(x)$

$$R_{2i}(x) \triangleq \begin{cases} \frac{1}{2\gamma_i(x)}I_{m_i}, & \beta_i(x)\ne 0\\ 0, & \beta_i(x)=0\end{cases} \tag{64}$$

and $\gamma_i(x)$ is given by (65) for all $i=1,\ldots,q$. See (65), as shown at the bottom of the page. Finally, $J(x_0,\phi(x(\cdot))) = \mathbf{e}^{\mathrm{T}}V(x_0)$, $x_0\in\mathbb{R}^n$, where $V:\mathbb{R}^n\to\overline{\mathbb{R}}_+^q$ is a control vector Lyapunov function for the dynamical system (54).

Proof: It follows from Theorem 5.1 that the feedback control law (57) globally asymptotically stabilizes the dynamical system (54) and the vector function $V:\mathbb{R}^n\to\overline{\mathbb{R}}_+^q$ is a control vector Lyapunov function for the dynamical system (54). Note that with (65) the feedback control law (57) can be rewritten as $\phi_i(x) = -\gamma_i(x)\beta_i(x)$, $x\in\mathbb{R}^n$, $i=1,\ldots,q$. Let the control vector Lyapunov function $V:\mathbb{R}^n\to\overline{\mathbb{R}}_+^q$ for (54) be a vector Lyapunov function candidate for (61). Then, the vector Lyapunov derivative components are given by

$$\begin{aligned}\dot{v}_i(x_i) &= v_i'(x_i)(f_i(x)+G_i(x)\sigma_i(\phi_i(x)))\\ &= \alpha_i(x) + \beta_i^{\mathrm{T}}(x)\sigma_i(\phi_i(x))\\ &\quad x\in\mathbb{R}^n, \qquad i=1,\ldots,q.\end{aligned} \tag{66}$$

Note that $\phi_i(x)=0$ and, hence, $\sigma_i(\phi_i(x))=0$ whenever $\beta_i(x)=0$ for all $i=1,\ldots,q$. In this case, it follows from (55) that $\dot{v}_i(x_i) < w_i(V(x))$, $x\in\mathbb{R}^n$, $\beta_i(x)=0$, $x\ne 0$, $i=1,\ldots,q$. Next, consider the case where $\beta_i(x)\ne 0$, $i=1,\ldots,q$. In this case, note that

$$\begin{aligned}&\alpha_i(x) - w_i(V(x)) - \frac{\gamma_i(x)}{2}\beta_i^{\mathrm{T}}(x)\beta_i(x)\\ &= \frac{-c_{0i}\beta_i^{\mathrm{T}}(x)\beta_i(x) + (\alpha_i(x)-w_i(V(x)))}{2}\\ &\quad - \frac{\sqrt{(\alpha_i(x)-w_i(V(x)))^2+(\beta_i^{\mathrm{T}}(x)\beta_i(x))^2}}{2}\\ &< 0, \qquad x\in\mathbb{R}^n, \quad \beta_i(x)\ne 0\end{aligned} \tag{67}$$

for all $i=1,\ldots,q$. Thus, the vector Lyapunov derivative componets given by (66) satisfy

$$\begin{aligned}&\dot{v}_i(x_i)\\ &< w_i(V(x)) + \frac{\gamma_i(x)}{2}\beta_i^{\mathrm{T}}(x)\beta_i(x) + \beta_i^{\mathrm{T}}(x)\sigma_i(\phi_i(x))\\ &= w_i(V(x)) + \frac{1}{2\gamma_i(x)}\phi_i^{\mathrm{T}}(x)\phi_i(x) - \frac{1}{\gamma_i(x)}\phi_i^{\mathrm{T}}(x)\sigma_i(\phi_i(x))\\ &= w_i(V(x)) + \frac{1}{\gamma_i(x)}\left[\frac{\phi_i^{\mathrm{T}}(x)\phi_i(x)}{2} - \phi_i^{\mathrm{T}}(x)\sigma_i(\phi_i(x))\right]\\ &\le w_i(V(x)), \qquad x\in\mathbb{R}^n, \qquad \beta_i(x)\ne 0\end{aligned} \tag{68}$$

$$\gamma_i(x) \triangleq \begin{cases} c_{0i} + \frac{(\alpha_i(x)-w_i(V(x))) + \sqrt{(\alpha_i(x)-w_i(V(x)))^2+(\beta_i^{\mathrm{T}}(x)\beta_i(x))^2}}{\beta_i^{\mathrm{T}}(x)\beta_i(x)} > 0, & \beta_i(x)\ne 0\\ 0, & \beta_i(x)=0\end{cases} \tag{65}$$

for all $\sigma_i(\cdot) \in \Phi_i$ and $i = 1, \ldots, q$. Since the dynamical system (48) is globally asymptotically stable it follows from Corollary 4.1 that the nonlinear dynamical system (61) is globally asymptotically stable for all $\sigma_i(\cdot) \in \Phi_i$, $i = 1, \ldots, q$.

To show that the feedback control law (57) minimizes (62) in the sense of (63), define the Hamiltonian

$$H(x,u) \triangleq \sum_{i=1}^{q} [L_{1i}(x) + u_i^{\mathrm{T}} R_{2i}(x) u_i + v_i'(f_i(x) + G_i(x) u_i)] \tag{69}$$

and note that $H(x, \phi(x)) = 0$ and $H(x,u) \geq 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, since $H(x,u) = \sum_{i=1}^{q} (u_i - \phi_i(x))^{\mathrm{T}} R_{2i}(x)(u_i - \phi_i(x))$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Thus

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_{t_0}^{\infty} \left[H(x(t), u(t)) - \sum_{i=1}^{q} v_i'(x(t))(f_i(x(t)) + G_i(x(t)) u_i(t)) \right] \mathrm{d}t \\ &= -\int_{t_0}^{\infty} \mathbf{e}^{\mathrm{T}} \dot{V}(x(t)) \mathrm{d}t + \int_{t_0}^{\infty} H(x(t), u(t)) \mathrm{d}t \\ &= -\lim_{t \to \infty} \mathbf{e}^{\mathrm{T}} V(x(t)) + \mathbf{e}^{\mathrm{T}} V(x_0) + \int_{t_0}^{\infty} H(x(t), u(t)) \mathrm{d}t \\ &\geq \mathbf{e}^{\mathrm{T}} V(x_0) \\ &= J(x_0, \phi(x(\cdot))) \end{aligned} \tag{70}$$

which yields (63). □

Remark 6.1: It follows from Theorem 6.1 that with the feedback stabilizing control law (57) the nonlinear dynamical system (54) has a sector (and hence gain) margin $(1/2, \infty)$ in each decentralized input channel. For details on stability margins for nonlinear dynamical systems, see [30] and [31].

Finally, note that Theorem 6.1 implies that

$$\alpha_i(x) - w_i(V(x)) - \theta_i \gamma_i(x) \beta_i^{\mathrm{T}}(x) \beta_i(x) \leq 0, \qquad x \in \mathbb{R}^n \tag{71}$$

for all $\theta_i \in [1/2, \infty)$ and $i = 1, \ldots, q$. Thus, if $\left|(\alpha_i(x) - w_i(V(x)))/\beta_i^{\mathrm{T}}(x)\beta_i(x)\right| \leq \mu_i$, $x \in \mathbb{R}^n$, $\beta_i(x) \neq 0$, $i = 1, \ldots, q$, then $|\gamma_i(x)| \leq c_{0i} + \mu_i + \mu_i\sqrt{1 + \mu_i^2} \triangleq \lambda_i$, $x \in \mathbb{R}^n$, $i = 1, \ldots, q$. In this case, the vector Lyapunov derivative components for the dynamical system (54) with the output $y = [y_1^{\mathrm{T}}, \ldots, y_q^{\mathrm{T}}]^{\mathrm{T}}$, where $y_i(x) \triangleq \beta_i(x)$, $x \in \mathbb{R}^n$, $i = 1, \ldots, q$, satisfy

$$\begin{aligned} \dot{v}_i(x_i) &= \alpha_i(x) + \beta_i^{\mathrm{T}}(x) u_i \\ &\leq w_i(V(x)) + \theta_i \gamma_i(x) \beta_i^{\mathrm{T}}(x) \beta_i(x) + \beta_i^{\mathrm{T}}(x) u_i \\ &\leq w_i(V(x)) + \theta_i \lambda_i y_i^{\mathrm{T}} y_i + y_i^{\mathrm{T}} u_i \\ &\quad x \in \mathbb{R}^n, \qquad u_i \in \mathbb{R}^{m_i}, \quad i = 1, \ldots, q. \end{aligned} \tag{72}$$

Inequality (72) imples that (54) is *exponentially vector dissipative* with respect to the *vector supply rate* $S(u,y) \triangleq [s_1(u_1,y_1), \ldots, s_q(u_q,y_q)]^{\mathrm{T}}$, where $s_i(u_i,y_i) = \theta_i \lambda_i y_i^{\mathrm{T}} y_i + y_i^{\mathrm{T}} u_i$, $i = 1, \ldots, q$, and with the control vector Lyapunov function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$ being a *vector storage function*. For details regarding vector dissipativity, see [22] and [32].

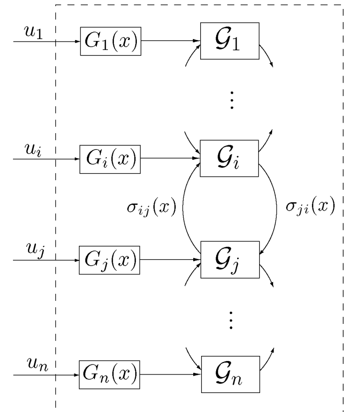

Fig. 1. Large-scale dynamical system $\mathcal{G}$.

VII. Decentralized Control for Large-Scale Nonlinear Dynamical Systems

In this section, we apply the proposed control framework to decentralized control of large-scale nonlinear dynamical systems [15]. Specifically, we consider the large-scale dynamical system $\mathcal{G}$ shown in Fig. 1 involving energy exchange between n interconnected subsystems. Let $x_i : [0, \infty) \to \overline{\mathbb{R}}_+$ denote the energy (and hence a nonnegative quantity) of the ith subsystem, let $u_i : [0, \infty) \to \mathbb{R}$ denote the control input to the ith subsystem, and let $\sigma_{ij} : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$, $i \neq j$, $i,j = 1, \ldots, n$, denote the instantaneous rate of energy flow from the jth subsystem to the ith subsystem.

An energy balance yields the large-scale dynamical system [33]

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(t_0) = x_0, \qquad t \geq t_0 \tag{73}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^{\mathrm{T}}$, $t \geq t_0$, $f_i(x) = \sum_{j=1, j\neq i}^{n} \phi_{ij}(x)$, where $\phi_{ij}(x) \triangleq \sigma_{ij}(x) - \sigma_{ji}(x)$, $x \in \overline{\mathbb{R}}_+^n$, $i \neq j$, $i,j = 1, \ldots, q$, denotes the net energy flow from the jth subsystem to the ith subsystem, $G(x) = \mathrm{diag}[G_1(x), \ldots, G_n(x)] = \mathrm{diag}[x_1, \ldots, x_n]$, $x \in \overline{\mathbb{R}}_+^n$, and $u(t) \in \mathbb{R}^n$, $t \geq t_0$. Here, we assume that $\sigma_{ij} : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$, $i \neq j$, $i,j = 1, \ldots, n$, are locally Lipschitz continuous on $\overline{\mathbb{R}}_+^n$, $\sigma_{ij}(0) = 0$, $i \neq j$, $i,j = 1, \ldots, n$, and $u = [u_1, \ldots, u_n]^{\mathrm{T}} : \mathbb{R} \to \mathbb{R}^n$ is such that $u_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, are bounded piecewise continuous functions of time. Furthermore, we assume that $\sigma_{ij}(x) = 0$, $x \in \overline{\mathbb{R}}_+^n$, whenever $x_j = 0$, $i \neq j$, $i,j = 1, \ldots, n$. In this case, $f(\cdot)$ is *essentially nonnegative* [33], [34] (i.e., $f_i(x) \geq 0$ for all $x \in \overline{\mathbb{R}}_+^n$ such that $x_i = 0$, $i = 1, \ldots, n$). The previous constraint implies that if the energy of the jth subsystem of $\mathcal{G}$ is zero, then this subsystem cannot supply any energy to its surroundings. Finally, in order to ensure that the trajectories of the closed-loop system remain in the nonnegative orthant of the state space for all nonnegative initial conditions, we seek a feedback control law $u(\cdot)$ that guarantees the closed-loop system dynamics are essentially nonnegative [34].

For the dynamical system $\mathcal{G}$, consider the control vector Lyapunov function candidate $V(x) = [v_1(x_1), \ldots, v_n(x_n)]^{\mathrm{T}}$, $x \in \overline{\mathbb{R}}_+^n$, given by

$$V(x) = [x_1, \ldots, x_n]^{\mathrm{T}}, \qquad x \in \overline{\mathbb{R}}_+^n. \tag{74}$$

Note that $V(0) = 0$ and $v(x) \triangleq \mathbf{e}^{\mathrm{T}} V(x)$, $x \in \overline{\mathbb{R}}_+^n$, is positive definite and radially unbounded. Furhtermore, consider the function

$$w(V(x), x) = \begin{bmatrix} -\sigma_{11}(v_1(x_1)) + \sum_{j=1, j\neq 1}^{n} \phi_{1j}(x) \\ \vdots \\ -\sigma_{nn}(v_n(x_n)) + \sum_{j=1, j\neq n}^{n} \phi_{nj}(x) \end{bmatrix}, \quad x \in \overline{\mathbb{R}}_+^n \tag{75}$$

where $\sigma_{ii} : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$, $i = 1, \ldots, n$, are positive definite functions, and note that $w(\cdot, x) \in \mathcal{W}$, $x \in \overline{\mathbb{R}}_+^n$, and $w(0, 0) = 0$. Also, note that it follows from Remark 5.2 that

$$\begin{aligned} \mathcal{R}_i &\triangleq \{x \in \overline{\mathbb{R}}_+^n,\, x_i \neq 0 : V_i'(x_i) G_i(x) = 0\} \\ &= \{x \in \overline{\mathbb{R}}_+^n,\, x_i \neq 0 : x_i = 0\} = \varnothing \end{aligned}$$

and, hence, condition (55) is satisfied for $V(\cdot)$ and $w(\cdot, \cdot)$ given by (74) and (75), respectively. To show that the dynamical system

$$\dot{z}(t) = w(z(t), x(t)), \quad z(t_0) = z_0, \qquad t \geq t_0 \tag{76}$$

where $z(t) \in \overline{\mathbb{R}}_+^n$, $t \geq t_0$, $x(t)$, $t \geq t_0$, is the solution to (73), and the ith component of $w(z, x)$ is given by $w_i(z, x) = -\sigma_{ii}(z_i) + \sum_{j=1, j\neq i}^{n} \phi_{ij}(x)$, $z \in \overline{\mathbb{R}}_+^n$, $x \in \overline{\mathbb{R}}_+^n$, is globally asymptotically stable with respect to z uniformly in x_0, consider the partial Lyapunov function candidate $\tilde{v}(z) = \mathbf{e}^{\mathrm{T}} z$, $z \in \overline{\mathbb{R}}_+^n$. Note that $\tilde{v}(\cdot)$ is radially unbounded, $\tilde{v}(0) = 0$, $\tilde{v}(z) > 0$, $z \in \overline{\mathbb{R}}_+^n$, $z \neq 0$, and

$$\begin{aligned} \dot{\tilde{v}}(z) &= -\sum_{i=1}^{n} \sigma_{ii}(z_i) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \phi_{ij}(x) \\ &= -\sum_{i=1}^{n} \sigma_{ii}(z_i) < 0 \end{aligned}$$

$z \in \overline{\mathbb{R}}_+^n$, $z \neq 0$. Thus, it follows from [21, Cor. 1] that the dynamical system (76), (73) is globally asymptotically stable with respect to z uniformly in x_0. Hence, it follows from Theorem 5.1 that $V(x)$, $x \in \overline{\mathbb{R}}_+^n$, given by (74) is a control vector Lyapunov function for the dynamical system (73).

Next, using (57) with $\alpha_i(x) = V_i'(x_i) f_i(x) = \sum_{j=1, j\neq i}^{n} \phi_{ij}(x)$, $\beta_i(x) = x_i$, $x \in \overline{\mathbb{R}}_+^n$, and $c_{0i} > 0$, $i = 1, \ldots, n$, we construct a globally stabilizing decentralized feedback controller for (73). It can be seen from the structure of the feedback control law that the closed-loop system dynamics are essentially nonnegative. Furthermore, since $\alpha_i(x) - w_i(V(x), x) = \sigma_{ii}(v_i(x_i))$, $x \in \overline{\mathbb{R}}_+^n$, $i = 1, \ldots, n$, this feedback controller is fully independent from $f(x)$ which represents the internal interconnections of the large-scale system dynamics, and hence, is robust against full modeling uncertainty in $f(x)$. Moreover, it follows from Theorem 6.1 and Remark 6.1 that the dynamical system (73) with the feedback stabilizing control law (57) has a sector (and hence gain) margin $(1/2, \infty)$ in each decentralized input channel, and hence, additionally guarantees robustness to multiplicative input uncertainty. Finally, the feedback controller minimizes the derived cost functional given by (62).

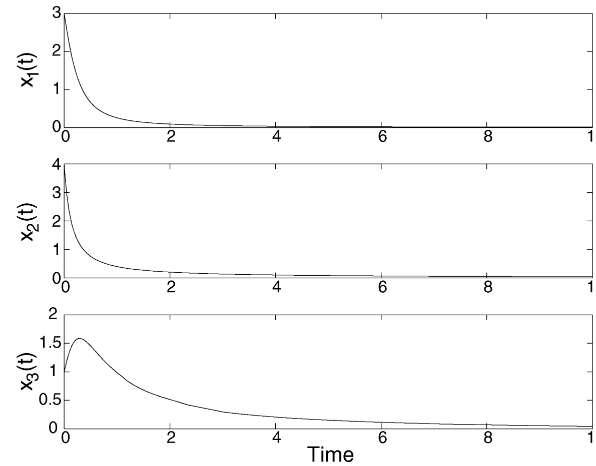

Fig. 2. Controlled system states versus time.

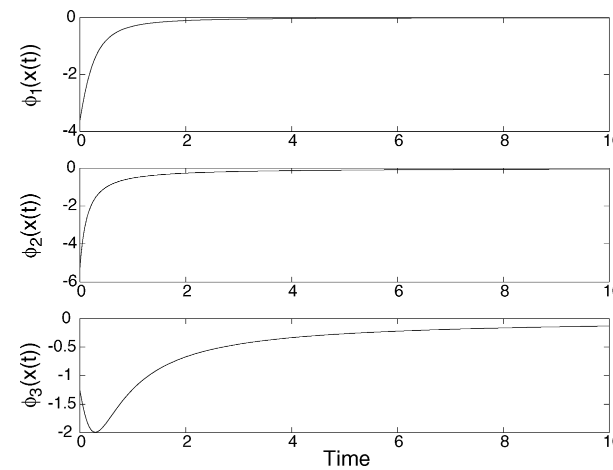

Fig. 3. Control signals in each decentralized control channel versus time.

For the following simulation we consider (73) with $\sigma_{ij}(x) = \sigma_{ij} x_i x_j$ and $\sigma_{ii}(x) = \sigma_{ii} x_i^2$, where $\sigma_{ij} \geq 0$, $i \neq j$, $i, j = 1, \ldots, n$, and $\sigma_{ii} > 0$, $i = 1, \ldots, n$. Note that in this case the conditions of Proposition 5.1 are satisfied, and hence, the feedback control law (57) is continuous on $\overline{\mathbb{R}}_+^n$. For our simulation we set $n = 3$, $\sigma_{11} = 0.1$, $\sigma_{22} = 0.2$, $\sigma_{33} = 0.01$, $\sigma_{12} = 2$, $\sigma_{13} = 3$, $\sigma_{21} = 1.5$, $\sigma_{23} = 0.3$, $\sigma_{31} = 4.4$, $\sigma_{32} = 0.6$, $c_{01} = 1$, $c_{02} = 1$, and $c_{03} = 0.25$, with initial condition $x_0 = [3, 4, 1]^{\mathrm{T}}$. Fig. 2 shows the states of the closed-loop system versus time and Fig. 3 shows control signal for each decentralized control channel as a function of time.

VIII. Conclusion

A generalized vector Lyapunov function framework for addressing stability of nonlinear dynamical systems was developed. In addition, a convergence result which specializes to the Krasovskii–LaSalle invariant set theorem for the case of a scalar comparison system was also presented. Moreover, the notion of a control vector Lyapunov function was introduced as a generalization of control Lyapunov functions, and its existence was shown to be equivalent to asymptotic stabilizability of a nonlinear dynamical system. Finally, the proposed control framework was used to construct decentralized controllers for large-scale nonlinear dynamical systems with robustness guarantees against full modeling uncertainty and multiplicative input uncertainty.

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