Joint density of eigenvalues in spiked multivariate models

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The classical methods of multivariate analysis are based on the eigenvalues of one or two sample covariance matrices. In many applications of these methods, for example, to high-dimensional data, it is natural to consider alternative hypotheses that are a low-rank departure from the null hypothesis. For rank 1 alternatives, this note provides a representation for the joint eigenvalue density in terms of a single contour integral. This will be of use for deriving approximate distributions for likelihood ratios and “linear” statistics used in testing. Copyright © 2014 John Wiley & Sons, Ltd.

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1 Introduction

The eigenvalues of one or two sample covariance matrices play a central role in multivariate analysis. A long list of examples, including principal components analysis (PCA), canonical correlations analysis, multivariate analysis of variance and multiple response linear regression are the main subject of many standard textbooks, such as Mardia et al. (1979) and Anderson (2003).

Under the common assumption of Gaussian data, much is known about the joint and marginal distribution of the eigenvalues. For example, under the typical null hypotheses, the joint density of the eigenvalues has an explicit formula, derived in 1939 in the celebrated and independent work of Fisher, Girshick, Hsu, Mood and Roy. Under general alternatives, the joint density is given by an integral over a group of matrices. If the number of variables, and hence eigenvalues, is large, $p$ say, as is common nowadays, this integral will be high dimensional, of dimension $O(p^2)$.

A remarkable classification of the joint density functions was given by James (1964), using hypergeometric functions of matrix argument. He showed how the classical multivariate methods could be organized into five cases, involving hypergeometric functions $_pF_q$ of different orders, specifically $0F_0$, $1F_0$, $1F_1$ and $2F_1$. Remarkable though this work is, and despite significant progress on the numerical computation of hypergeometric functions, for example, Koev & Edelman (2006), these expressions for the joint densities have proved challenging to work with in application.

In many high-dimensional applications, however, it may be reasonable to consider alternative hypotheses that are low-rank departures from the null. For some examples, see Johnstone & Nadler (2013). In this note, we consider the simplest case, namely rank 1 deviations, and show that the joint eigenvalue density can then be reduced to a single (contour) integral.

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We believe this integral representation to be of interest at least because it is amenable to approximation when dimension \( p \) is large, leading to simple approximations to at least certain aspects of these multivariate eigenvalue distributions.

We mention two examples of such applications.

(i) Derivation of limiting Gaussian approximations for “linear statistics” (including, for example, the likelihood ratio test, and “high-dimension-corrected” likelihood ratio test, Onatski et al. (2013) and Wang et al. (2013)). Particular cases \( _0F_0, _0F_1, _1F_1 \) have been given for complex data by Passemier et al. (2014a).

(ii) Delineation of the region of contiguous alternatives to the null hypothesis and description of the Gaussian limit for the log-likelihood ratio process inside the contiguity region. This leads to a comparative understanding of the power properties of various hypothesis tests, both traditional and new, in the contiguity region. This example has been studied in the case of PCA, corresponding to \(_0F_0\), by Onatski et al. (2013), and work is in progress to apply the result of this note to the general \( _pF_q \) cases.

We will adopt James’s systematization in order to give a unified derivation of our contour formulas. We give the rank 1 formula for \( _pF_q \) in real and complex cases, Section 2. This can be converted directly into an expression for the joint density function for the eigenvalues in each of James’s five cases (for both \( \mathbb{R} \) and \( \mathbb{C} \)). Section 3 illustrates this process in one case, testing equality of covariance matrices, for real data (i.e. \( _1F_0 \)).

In the real case, the proof of Section 2 applies only to even dimension \( p \). Section 4 gives a different proof valid for all integer \( p \).

## 2 Contour integral representation for rank 1

Let \( X, Y \) be \( r \times r \) Hermitian matrices. The definitions of hypergeometric functions with one and two matrix arguments are given, for example, by James (1964), with separate expressions for real and complex cases.

The definitions simplify in our special case in which \( X \) has rank 1, with non-zero eigenvalue \( x \). For \( a \in \mathbb{C} \), let \((a)_k = a(a + 1) \cdots (a + k - 1)\), \((a)_0 = 1\) be the rising factorial, and for vectors of parameters \( a = (a)_p \), \( b = (b)_q \), with \( a_i \in \mathbb{C} \) and \( b_i \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \), adopt the abbreviation

\[
\rho_k = \rho_k(a, b) = \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k}.
\]

If \( X \) has rank 1 as described, define

\[
_pF_q^\alpha(a, b; X, Y) = \sum_{k=0}^{\infty} \rho_k \frac{(1/\alpha)_k x^k C_k^\alpha(Y)}{(r/\alpha)_k k!}.
\] (1)

Here, \( \alpha > 0 \) indexes a one parameter family that includes the real \( (\alpha = 2) \) and complex \( (\alpha = 1) \) cases. Also, \( C_k^\alpha \) are Jack polynomials (e.g. Macdonald (1995)): in the real case \( (\alpha = 2) \), they reduce to James’s zonal polynomials (e.g. Muirhead (1982)), and in the complex case \( (\alpha = 1) \), to a normalization of the Schur functions (e.g. Dumitriu et al. (2007)). A contour formula for \( C_k^\alpha(Y) \) is quoted later; for now, we note that \( C_k^\alpha(X) = x^k \), and (e.g. Wang (2012, eq. (245))) that
The main result of this note can now be stated.

**Proposition 1**

Suppose that \( p \leq q + 1 \), \( X \) is rank 1 with positive eigenvalue \( x \) and that \( Y \) is positive definite with eigenvalues \( (y_j)_j = 1 \).

(i) Suppose that \( r/\alpha \) is a positive integer, say \( r/\alpha = m + 1 \), and that \( a_i \notin \{1, \ldots, m\} \) and \( b_i \notin \{m, m-1, m-2, \ldots\} \). Then,

\[
pFqalpha(a, b; x, Y) = \frac{\Gamma(m + 1)}{x^m \rho_m} \frac{1}{2\pi i} \int_{\Gamma} pFq(a - m, b - m; xs) \prod_{j=1}^{r} \frac{1}{(s - y_j)^{1/\alpha}} ds,
\]

where the contour \( \Gamma \) starts at \(-\infty\), encircles 0 and \( (y_j) \) counterclockwise and returns to \(-\infty\). Further, \( a - m \) denotes the vector with entries \( a_i - m \) and

\[
\rho_m' = \rho_m(a - m, b - m).
\]

Equality holds in the common domain of analyticity of both sides: \( \mathbb{C} \) if \( p \leq q \) and \( \mathbb{C} \setminus (1, \infty) \) if \( p = q + 1 \).

(ii) If instead \( r/\alpha = m + \epsilon \) for \( \epsilon \in (0, 1) \) and non-negative integer \( m \), then under the same conditions

\[
pFqalpha(a, b; x, Y) = \frac{(e)_m}{x^m \rho_m'} \frac{1}{2\pi i} \int_{\Gamma} s^{r-1} pFq+1(a - m, 1, b - m, \epsilon; xs) \prod_{j=1}^{r} \frac{1}{(s - y_j)^{1/\alpha}} ds.
\]

(iii) If \( \alpha = 2 \), then formula (2) holds for any integer \( r \), still with \( m + 1 = r/2 \), if the symbol \( (a)_m \) is interpreted as \( \Gamma(a + m) / \Gamma(a) \) for non-integer \( m \).

Thus, in the real (\( \alpha = 2 \)) and complex (\( \alpha = 1 \)) cases of most interest in applications, formula (2) holds for all positive integer \( r \).

Particular cases of (2) are already known: \( _0F_0 \) for both real and complex cases (Mo, 2012; Onatski et al., 2013), for general \( \alpha \), Wang (2012) and Forrester (2011), and for the complex case only, \( _0F_1 \) (Dharmawansa, 2013) and \( _1F_1 \) (Passemier et al., 2014a). Wang (2012) also gives formula (3) in the \( _0F_0 \) case. A generalization of (i) to the multi-spike case has been given for \( _0F_0 \) by Onatski (2014) and recently extended to \( pFq \) by Passemier et al. (2014b).
Proof

Parts (i) and (ii) are shown here; part (iii) uses a different argument and is deferred to Section 4. We begin with a result from Wang (2012, eq. (248)), which states that

\[
(1/\alpha)_k \frac{C_k(Y)}{k!} = \frac{1}{2\pi i} \int_{\Gamma'} \prod_{j=1}^{r} \frac{1}{(1 - zy_j)^{1/\alpha} z^{k+1}} \, dz.
\]

Here, the contour \( \Gamma' \) encircles zero and is chosen small enough so that all \( y_j^{-1} \) lie outside.

Insert this into (1) and interchange summation and integration to obtain

\[
\rho F_q^x(a, b; X, Y) = \frac{1}{2\pi i} \int_{\Gamma'} \prod_{j=1}^{r} \frac{1}{(1 - zy_j)^{1/\alpha}} G(z; x) \, dz,
\]

where the series

\[
G(z; x) = \sum_{k=0}^{\infty} \frac{\rho_k}{(r/\alpha)_k} \frac{x^k}{z^{k+1}}
\]

converges for all \( x, z \) if \( p \leq q \) and for \( |x/z| < 1 \) if \( p = q + 1 \).

Now, write \( r/\alpha = m + 1 \) and introduce the variable \( l = k + m \), so that

\[
G(z; x) = \sum_{l=m}^{\infty} \frac{\rho_{l-m}}{(m + 1)l_{l-m}} \frac{x^{l-m}}{z^{l+1}} = \frac{m!}{x^m} z^{m-1} \sum_{l=m}^{\infty} \frac{\rho_l(a - m, b - m)}{l!} \left( \frac{x}{z} \right)^l,
\]

where we have used \( (m + 1)l_{l-m} = l! / m! \) and noted that \( (c)_{l-m} = (c - m)_{l/(c - m)} \) so that

\[
\rho_{l-m}(a, b) = \frac{\rho_l(a - m, b - m)}{\rho_m(a - m, b - m)}.
\]

Let \( G_0(z; x) \) denote the function obtained by extending the summation in (5) down to \( l = 0 \), so that

\[
G_0(z; x) = \frac{m!}{x^m} z^{m-1} \frac{\rho_F^x(a - m, b - m; x/z)}{\rho_m(a - m, b - m)}.
\]

As we are adding a polynomial to \( G \) and a term that is analytic within the contour in (5), the value of the integral is unchanged. Hence,

\[
\rho F_q^x(a, b; X, Y) = \frac{m!}{x^m} \frac{1}{\rho_m} \frac{1}{2\pi i} \int_{\Gamma'} \prod_{j=1}^{r} \frac{z^{m-1}}{(1 - zy_j)^{1/\alpha}} eF_q(a - m, b - m; x/z) \, dz.
\]

The change of variables \( z = 1/s \) yields

\[
\frac{1}{2\pi i} \int_{\Gamma'} \frac{z^{m-1}}{\prod (1 - zy_j)^{1/\alpha}} F(x/z) \, dz = \frac{1}{2\pi i} \int_{\Gamma'} \frac{1}{s^{m+1}} \prod (1 - y_j/s)^{1/\alpha} \, ds,
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma} \frac{F(xs)}{\prod (s - y_j)^{1/\alpha}} \, ds,
\]
where the image $\Gamma''$ of $\Gamma$ is deformed to $\Gamma$ as described in the proposition statement in order to avoid the branch cut in the final formula. Here, we use the analytic continuations of $pFq$; entire for $\rho \leq q$ and for $\rho = q + 1$ analytic off the positive real axis $(1, \infty)$. The result follows.

When $r/\alpha = m + \epsilon$, we modify the argument. In (5), replace $(m+1)_{-m}$ by $(m+\epsilon)_{-m} = (\epsilon)_i/(\epsilon)_m$ to obtain

\[
G(z; x) = \frac{(\epsilon)_m z^{m-1}}{\rho_m} \sum_{l=m}^{\infty} \frac{\rho(a-m, b-m)(1)_l}{(\epsilon)_l} \frac{1}{l!} \left( \frac{x}{z} \right)^l.
\]

Proceeding as before, and extending the summation to $l = 0$, so that

\[
G_0(z; x) = \frac{(\epsilon)_m z^{m-1}}{\rho_m} p_{1Fq+1}(a-m, b-m, \epsilon; x/z),
\]

we obtain formula (3).

\section{Example}

Consider the problem of testing equality of covariance matrices—the $1F0$ case in James (1964). Thus, suppose that $n_1, n_2 \geq p$ and that $p \times n_1$ and $p \times n_2$ real data matrices $X = \{X_1, \ldots, X_{n_1}\}$ and $Y = \{Y_1, \ldots, Y_{n_2}\}$ have columns $X_\gamma, Y_\gamma$ with mean zero and covariance matrices $\Sigma_1$ and $\Sigma_2$, respectively. A signal detection application is described in Johnstone & Nadler (2013, Sec. 3).

Suppose that the observation vectors are independent Gaussian, so that $A_1 = XX'$ and $A_2 = YY'$ have Wishart distributions $W_p(n_1, \Sigma_1)$ and $W_p(n_2, \Sigma_2)$, respectively. Then James (1964, eq. (65)) gives an expression for the joint density of the eigenvalues ($f_j$) of $A_1 A_2^{-1}$. To state it, we introduce notation $|A| = \det(A), F = \text{diag}(f_j)$ and $\Delta = \Sigma_1 \Sigma_2^{-1}$. We transform this expression, following Muirhead (1982, pp. 313–314) to obtain for $n = n_1 + n_2$ and $f_1 > f_2 > \cdots > f_p$,

\[
p(f; \Delta) = \frac{c_{p,n_1,n_2}}{|\Delta|^{n_1/2} |l + F|^{n_2/2} - 1F_0 \left( \frac{n}{2}; l - \Delta^{-1}, F(l + F)^{-1} \right)} \prod_{j<p} (f_j - f_p),
\]

where in this real case, $\alpha = 2$, we have written $1F_0$ for $1F^0_0$. The normalization constant is given in terms of the multivariate Gamma function (Muirhead, 1982, p. 61) by

\[
c_{p,n_1,n_2} = \frac{\pi^{p^2/2} \Gamma_p \left( \frac{1}{2} n_1 \right)}{\Gamma_p \left( \frac{1}{2} n_1 \right) \Gamma_p \left( \frac{1}{2} n_2 \right)}.
\]

In the spirit of application (ii) in the Introduction, we may consider the likelihood ratio for testing the null hypothesis that $\Sigma_1 = \Sigma_2$. Writing $\Lambda = F(l + F)^{-1}$, we have

\[
L(\Delta; \Lambda) = \frac{p(\Lambda; \Delta)}{p(\Lambda; l)} = |\Delta|^{-n_1/2} 1F_0 \left( \frac{n}{2}; l - \Delta^{-1}, \Lambda \right).
\]

Turning now to apply the result of this paper, suppose that $\Sigma_1$ is a rank 1 perturbation of $\Sigma_2$, so that $\Sigma_1 = (I + \psi h \psi') \Sigma_2$ for real $h$ and for $\psi$ a unit vector in $\mathbb{R}^p$. In this case, $\Delta = l + \psi h \psi'$, so that $l - \Delta^{-1}$ has rank 1, with non-zero eigenvalue $\tau = h/(1 + h)$.
As all components of $\Lambda = F(l + F)^{-1}$ are less than one, we may apply the contour formula (2). As $F_0(a; x) = (1-x)^{-a}$, we obtain

$$L(\tau; \Lambda) = \frac{n - p}{2} B \left( \frac{n - p}{2} \right) \frac{(1 - \tau)^{n/2}}{\tau^{p/2-1}} \frac{1}{2\pi i} \int \left( 1 - \tau s \right)^{-2n-p+2} \frac{1}{\prod_j (s - \lambda_j)^{1/2}} ds,$$

where $B(\alpha, \beta)$ is the usual beta function. This is a form suitable for asymptotic approximation, the details of which will be reported elsewhere.

**Remark**
A useful check on this last formula is obtained by letting the error degrees of freedom $n_2 \to \infty$ while keeping $p$ and $n_1$ fixed. This limit corresponds to the case where $\Sigma_2$ is known, say $\Sigma_2 = I$ for convenience here, and we consider the single matrix rank 1 model $\Sigma_1 = I + \psi h \psi^t$ and test the hypothesis that $h = 0$. To compare with the formula of Onatski et al. (2013, Lemma 3), let $(\mu_j)$ be the eigenvalues of $n_1^{-1} A_1 (n_2^{-1} A_2)^{-1}$, so that $\lambda_j = f_j/(1 + f_j) = n_1 \mu_j/(n_2 + n_1 \mu_j)$. With the change of variables $s = n_1 z/n_2$, the previous display converges to

$$L(\tau; \mu) = \Gamma \left( \frac{p}{2} \right) \left( \frac{2}{n_1} \right)^{p/2-1} \frac{(1 - \tau)^{n/2}}{\tau^{p/2-1}} \frac{1}{2\pi i} \int \frac{e^{n_1 z/2} \prod_j (z - \mu_j)^{-1/2}}{2} dz,$$

which is the cited expression for the $\alpha F_0$ likelihood ratio.

### 4 Real case, integer $r$

Here, we prove Proposition 1, for real matrices with integer dimension $r$, not necessarily even. A similar result, with proof extending that of Onatski et al. (2013, Lemma 2), has been obtained by Alexei Onatski (personal communication) and will appear elsewhere.

Our goal is to prove the validity of the following expression for $0 \leq p \leq q + 1$:

$$\rho^2 F_q^2(a, b; X, Y) = \frac{\Gamma(m + 1)}{\chi^m \rho_m} \frac{1}{2\pi i} \int \rho F_q(a - m, b - m; x) \Delta_y(s) ds \quad (7)$$

where we have defined $\Delta_y(s) = \prod_{j=1}^r (s - y_j)^{-1/2}$. The contour $\Gamma$ starts from $-\infty$ and encircles $y_1, y_2, \ldots, y_r$ in the positive direction (i.e., counterclockwise) and goes back to $-\infty$.

In what follows, we provide an inductive proof for the preceding claim. First, we establish the initial cases: $\alpha F_q$ for $q \geq 0$ and, separately, $\alpha F_0$. The inductive step establishes truth for $\rho+1 F_{q+1}$ given truth for $\rho F_q$. Also, it is worth mentioning that we assume all powers have their principal values and all angles in the range $[-\pi, \pi]$.

The following alternative representation of the hypergeometric function of two matrix arguments is useful in the sequel. Let $O(r)$ be the orthogonal group, and let $(dQ)$ be the invariant measure on $O(r)$ normalized to make the total measure unity. Then, following James (1964), we can write

$$\rho^2 F_q^2(a, b; X, Y) = \int_{O(r)} \rho^2 F_q^2(a, b; XQ'YQ) (dQ). \quad (8)$$
Moreover, let us assume, without loss of generality, that \( Y = \text{diag}(y_1, y_2, \ldots, y_r) \). As \( X \) is rank 1, we can further simplify (8) to yield

\[
\rho F^2_q(a, b; X, Y) = \int_{S(r)} \rho F_q(a, b; xq_i', Y_i) \, (dq_i),
\]

where \( S(r) \) is the \( r-1 \) dimensional sphere embedded in \( \mathbb{R}^r \), \( q_i \) is the first column of \( Q \) and \((dq_i)\) is the invariant measure on \( S(r) \) normalized such that the total measure is one.

4.1. Initial cases

We first show that the statement (7) is true for \( \rho F_q \). With the standard notation \( \rho F_q(b; z) = \rho F_q(b; z)/\prod_{j=1}^q \Gamma(b_j) \), this is equivalent to showing that, for \( q \geq 0 \),

\[
\rho F_q(b; X, Y) = \frac{\Gamma(m + 1)}{\chi^m} \frac{1}{2\pi i} \int_{\Gamma} \rho F_q(b - m; xs) \Delta_{q_j}(s) \, ds. 
\]

Our tool is a contour representation of Erdélyi (1937, eq. (7.4)):

\[
\rho F_q(b; z) = \frac{1}{(2\pi i)^q} \int_{-\infty}^{(0+)} \cdots \int_{-\infty}^{(0+)} \exp \left( \sum_{j=1}^q w_j + \frac{z}{\prod_{j=1}^q w_j} \right) \prod_{j=1}^q \frac{dw_j}{w_j^b},
\]

where each contour starts from \(-\infty\) and encircles the origin in the positive sense and goes back to \(-\infty\). We use multi-index notation \( w^b = \prod w_j^b, w = \prod w_j \) and \( dw = \prod dw_j \).

We use the spherical average (9), then Erdélyi's representation, and change order of integration, to obtain

\[
\rho F_q(b; X, Y) = \int_{S(r)} \rho F_q(b; xq_i', Y_i) \, (dq_i) 
\]

\[
= \frac{1}{(2\pi i)^q} \int_{-\infty}^{(0+)} \cdots \int_{-\infty}^{(0+)} \exp \left( \sum_{j=1}^q w_j \right) \int_{S(r)} \exp \frac{\bar{x}_s q_j y_i}{w_j} \Delta_{q_j}(s) \, ds \, dw. 
\]

A change of variable in Onatski et al. (2013, Lemma 2) shows that for \( x, w > 0 \),

\[
\int_{S(r)} \exp \frac{\bar{x}_s q_j y_i}{w_j} \Delta_{q_j}(s) \, ds = \frac{\Gamma(r/2)}{2\pi i} \left( \frac{w}{x} \right)^{r/2 - 1} \int_{\Gamma} \exp \frac{\bar{x}_s}{w} \Delta_{q_j}(s) \, ds,
\]

and the equality extends by analyticity to all non-zero \( w \in \mathbb{C} \). Inserting this integral in (12) and noting that \( \frac{r}{2} = m + 1 \), we obtain

\[
\rho F_q(b; X, Y) = \frac{\Gamma(m + 1)}{\chi^m (2\pi i)^{q+1}} \int_{-\infty}^{(0+)} \cdots \int_{-\infty}^{(0+)} \exp \sum_{j=1}^q w_j \int_{\Gamma} \exp \frac{\bar{x}_s}{w} \Delta_{q_j}(s) \, ds \, dw. 
\]

Finally, we change the order of integration and again make use of (11) to arrive at the desired equality (10). This proves the validity of the statement (7) for \( p = 0 \).
Now, we show that, for \( x \max \{y_j\} < 1 \),

\[
1F_0^2(a; X, Y) = \frac{\Gamma(m+1)}{x^m} \frac{1}{2\pi i} \int_\Gamma 1F_0(a-m; xs) \Delta_y(s) \, ds.
\]  

(15)

We use identity (9), the special form \( 1F_0(a; z) = (1-z)^{-a} \) and the relation

\[
\frac{1}{s^a} = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-st} \, dt, \quad \Re(s) > 0, \Re(a) > 0
\]  

(16)

to obtain, after observing that \( x \max \{y_j\} < 1 \) implies \( xq'_r y_q < 1 \),

\[
1F_0^2(a; X, Y) = \int_{S(t)} \frac{1}{(1-xq'_r y_q)^s} (dq_r) = \int_0^\infty t^{a-1} e^{-t} \int_{S(t)} e^{xq'_r y_q} (dq_r) \, dt.
\]  

(17)

Now, substitute the contour identity (14) with \( t = 1/w \), and with the contour chosen to encircle \( \{y_j\} \) and to lie to the left of \( 1/x \). We obtain

\[
1F_0^2(a; X, Y) = \frac{\Gamma(r/2)}{\Gamma(a)x^{\frac{r}{2}-1}} \frac{1}{2\pi i} \int_0^\infty \int_\Gamma t^{\frac{r}{2}-\frac{1}{2}} e^{-t(1-\frac{1}{x})} \Delta_y(s) \, ds \, dt
\]

valid for \( \Re(a) > \frac{r}{2} - 1 \), after changing order of integration and using (16) and the fact that \( \Re(s) < 1/x \). Recalling that \( m = r/2 - 1 \) and \( 1F_0(a; z) = (1-z)^{-a} \), the final form reduces to the right-hand side of (15), under the condition \( \Re(a) > m \). However, both sides of the preceding equality, which we have established only in the domain \( \Re(a) > m \) of complex plane, are analytic functions. Therefore, the equality must hold in the whole region of the analyticity of \( a \). This establishes the claim (15).

4.2. Inductive step

First, some notation. We write \( a_+ = (a_1, a_2, \ldots, a_p) \) and \( b_+ = (b_1, b_2, \ldots, b_q) \) for the augmentations of \( a \) and \( b \), and abbreviate \( p+F_{q+1} \) by \( p+F_{q+} \). Thus, the induction step amounts to establishing the validity of the following statement, given the statement (7) is true

\[
p+F^2_{q+}(a_+, b_+; X, Y) = \frac{\Gamma(m+1)}{x^m} \frac{1}{2\pi i} \int_{p+F_{q+}} (a_+ - m, b_+ - m; xs) \Delta_y(s) \, ds
\]  

(18)

where

\[
\rho_{m+} = \frac{\Gamma(a) \Gamma(b + m)}{\Gamma(a - m) \Gamma(a + m)}
\]  

(19)

We use a reparametrized version of the beta density

\[
\phi(t; \alpha, \beta) = \frac{\Gamma(\beta)}{\Gamma(\alpha) \Gamma(\beta - \alpha)} t^{\alpha-1}(1-t)^{\beta-\alpha-1},
\]
and the integral representation of the generalized hypergeometric function (Erdélyi, 1937, eq. (3.2))

\[ p + F_q (a_+, b_+; x) = \int_0^1 \phi(t; \alpha, \beta) p F_q (a, b; xt) \, dt, \]  

(20)

where \( \Re(\beta) > \Re(\alpha) > 0 \), along with (9), in order to write the left side of (18) as

\[ p + F_q^2 (a_+, b_+; X, Y) = \int_{S(\rho)} p + F_q (a_+, b_+; xq Y_q) \, (dq_r) \]

\[ = \int_{S(\rho)} \int_0^1 \phi(t; \alpha, \beta) p F_q (a, b; xt q Y_q) \, dt \, (dq_r) \]

\[ = \int_0^1 \phi(t; \alpha, \beta) \int_{S(\rho)} p F_q (a, b; xt q Y_q) \, (dq_r) \, dt \]

\[ = \int_0^1 \phi(t; \alpha, \beta) p F_q^2 (a, b; tX, Y) \, dt, \]

where we have changed the order of integration and again used (9). The final expression can be rewritten with the help of our induction hypothesis (7) as

\[ \Gamma(m + 1) \frac{1}{x^m \rho_m} \int_0^1 t^{-m} \phi(t; \alpha, \beta) \int \rho F_q (a - m, b - m; xt) \, \Delta_Y(s) \, ds \, dt. \]  

(21)

Now, use the identity

\[ t^{-m} \phi(t; \alpha, \beta) = \phi(t; \alpha - m, \beta - m) \frac{\Gamma(\beta) \Gamma(\alpha - m)}{\Gamma(\beta - m) \Gamma(\alpha)} \]

and note from (19) that the ratio of Gamma functions equals \( \rho_m / \rho_m^* \). Inserting this into (21) and changing the order of integration, we obtain

\[ \Gamma(m + 1) \frac{1}{x^m \rho_m^*} \int \Delta_Y(s) \int_0^1 \phi(t; \alpha - m, \beta - m) p F_q (a - m, b - m; xt) \, dt \, ds. \]

Now, again use (20), along with the restriction \( \Re(\alpha) > m \), to yield (18) in the domain \( \Re(\beta) > \Re(\alpha) > m \) of \( \mathbb{C} \). As both sides of equality (18) are analytic functions, the equality must hold in the whole region of the analyticity of \( \alpha \) and \( \beta \). This completes the induction step.

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**References**


