Radiation Fields and Patterns

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For the magnetic vector potential, we obtained

$$\nabla^2 A - \mu \varepsilon \frac{\partial^2 A}{\partial t^2} = -\mu J$$

(1)

which reduces to

$$\nabla^2 A_z - \mu \varepsilon \frac{\partial^2 A_z}{\partial t^2} = -\mu J_z$$

(2)

for \( A = A_z i_z \) and \( J = J_z i_z \).
Fields due to Hertzian Dipole

We expressed $A$ in terms of its components in spherical coordinates,

$$A = \frac{\mu I_0 dl \cos(\omega t - \beta r)}{4\pi r} (\cos \theta i_r - \sin \theta i_\theta) \quad (3)$$
The magnetic field due to the Hertzian dipole is then given by

\[
H = \frac{B}{\mu} = \frac{1}{\mu} \nabla \times A
\]

\[
= \frac{1}{\mu} \begin{vmatrix}
  i_r & i_\theta & i_\phi \\
  r^2 \sin \theta & r \sin \theta & r \\
  A_r & r A_\theta & 0 \\
\end{vmatrix} = \frac{1}{\mu r} \left[ \frac{\partial}{\partial r} (rA_\theta) - \frac{\partial A_r}{\partial \theta} \right]
\]
or

\[ H = \frac{I_0 dl \sin \theta}{4\pi} \left[ \frac{\cos(\omega t - \beta r)}{r^2} - \frac{\beta \sin(\omega t - \beta r)}{r} \right] \hat{i}_\phi \]  (4)
Using Maxwell’s curl equation for $H$ with $J$ set equal to zero in view of perfect dielectric medium, we then have

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon} \nabla \times H$$

$$= \frac{1}{\varepsilon} \left| \begin{array}{ccc}
    i_r & i_\theta & i_\phi \\
    \frac{r^2 \sin \theta}{\partial} & \frac{r \sin \theta}{\partial} & \frac{r}{\partial} \\
    \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\
    0 & 0 & r \sin \theta H_\phi
\end{array} \right|$$

$$= \frac{1}{\varepsilon r^2 \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta H_\theta) i_r - \frac{1}{\varepsilon r \sin \theta} \frac{\partial}{\partial r} (r \sin \theta H_\phi) i_\theta$$
or

\[ E = \frac{2I_0 dl \cos \theta}{4\pi \varepsilon \omega} \left[ \sin(\omega t - \beta r) \frac{1}{r^3} + \frac{\beta \cos(\omega t - \beta r)}{r^2} \right] i_r 
+ \frac{I_0 dl \sin \theta}{4\pi \varepsilon \omega} \left[ \sin(\omega t - \beta r) \frac{1}{r^3} + \frac{\beta \cos(\omega t - \beta r)}{r^2} \right] i_\theta 
- \frac{\beta^2 \sin(\omega t - \beta r)}{r} \right] i_\theta. \] (5)

Equations 5 and 4 represent the electric and magnetic fields, respectively, due to the Hertzian dipole.

\[ \]
The following observations are pertinent to these field expressions:

- They satisfy the two Maxwell’s curl equations. In fact, we have obtained 5 from 4 by using the curl equation for $H$. We can verify that 4 follows from 5 through the curl equation for $E$.

- They contain terms involving $1/r^3$, $1/r^2$, and $1/r$. Far from the dipole such that $\beta r \gg 1$, the $1/r^3$ and $1/r^2$ terms are negligible compared to the $1/r$ terms.

Thus the time-average Poynting vector varies proportionately to $1/r^2$ and is directed entirely in the radial direction. This is consistent with the physical requirement that for the time-average power crossing all possible spherical surfaces centered at the dipole to be the same, the power density must be inversely proportional to $r^2$, since the surface areas of the spherical surfaces are proportional to the squares of their radii.
For $\beta \ll 1$, the $1/r^3$ terms dominate the $1/r^2$ terms which in turn dominate the $1/r$ terms. Also, $\sin(\omega t - \beta r) \approx (\sin \omega t - \beta r \cos \omega t)$ and $\cos(\omega t - \beta r) \approx (\cos \omega t + \beta r \sin \omega t)$, so that

$$E \approx \frac{I_0 dl \sin \omega t}{4\pi \varepsilon \omega r^3} (2 \cos \theta i_r + \sin \theta i_\theta) \quad (6)$$

$$H \approx \frac{I_0 dl \cos \omega t}{4\pi r^2} \sin \theta i_\phi \quad (7)$$
Equation 7 gives the same $B$ as the magnetic field given by Biot-Savart law applied to a current element $Idli_z$ at the origin and then $I$ replaced by $I_0 \cos \omega t$, that is, $I(t)$.

Thus electrically close to the dipole, where retardation effects are negligible, the field expressions approach toward the corresponding static field expressions with the static source terms simply replaced by the time varying source terms.
Example 2.1
Let us consider in free space a Hertzian dipole of length 0.1 m situated at the origin and along the $z$-axis, carrying the current $10 \cos 2\pi \times 10^7 t$ A. We wish to obtain the electric and magnetic fields at the point $(5, \pi/6, 0)$.

For convenience in computation of the amplitudes and phase angles of the field components, we shall express the field components in phasor form. Thus replacing $\cos(\omega t - \beta r)$ by $e^{-j\beta r}$ and $\sin(\omega t - \beta r)$ by $-je^{-j\beta r}$, we have
\[
\vec{E}_r = \frac{2I_0dl \cos \theta}{4\pi \varepsilon \omega} \left( -\frac{j}{r_3} + \frac{\beta}{r^2} \right) e^{-j\beta r} \\
= \frac{2\beta^2 \eta I_0 dl \cos \theta}{4\pi} \left[ -j \frac{1}{(\beta r)^3} + \frac{1}{(\beta r)^2} \right] e^{-j\beta r}
\] (8)

\[
\vec{E}_\theta = \frac{I_0 dl \sin \theta}{4\pi \varepsilon \omega} \left( -\frac{j}{r_3} + \frac{\beta}{r^2} + \frac{j\beta^2}{r} \right) e^{-j\beta r} \\
= \frac{\beta^2 \eta I_0 dl \sin \theta}{4\pi} \left[ -j \frac{1}{(\beta r)^3} + \frac{1}{(\beta r)^2} + j \frac{1}{\beta r} \right] e^{-j\beta r}
\] (9)
\[ \bar{H}_\phi = \frac{I_0 dl \sin \theta}{4\pi} \left( \frac{1}{r^2} + \frac{j\beta}{r} \right) e^{-j\beta r} \]

\[ = \beta_2 I_0 dl \sin \theta \left[ \frac{1}{(\beta r)^2} + j \frac{1}{\beta r} \right] e^{-j\beta r} \]  \hspace{1cm} (10)

where \( \eta = \sqrt{\mu/\varepsilon} \) is the intrinsic impedance of the medium. Using \( I_0 = 10 \) A, \( dl = 0.1 \) m, \( f = 10^7 \) Hz, \( \mu = \mu_0 \), \( \varepsilon = \varepsilon_0 \), \( r = 5 \) m, and \( \theta = \pi/6 \), and carrying out the computations with the aid of a computer we obtain

\[ \bar{E}_r = 2.8739 \angle -103.679^\circ \text{ V/m} \]
\[ \vec{E}_\theta = 0.6025 \angle -54.728^\circ \text{ V/m} \]
\[ \vec{H}_\phi = 0.0023 \angle -13.679^\circ \text{ A/m} \]

Thus the required fields are

\[ E = 2.8739 \cos(2\pi \times 10^7 t - 0.576\pi) i_r \]
\[ + 0.6025 \cos(2\pi \times 10^7 t - 0.304\pi) i_0 \text{ V/m} \]
\[ H = 0.0023 \cos(2\pi \times 10^7 t - 0.076\pi) i_\phi \text{ A/m} \]
In the preceding section we derived the expressions for the complete electromagnetic field due to the Hertzian dipole. These expressions look very complicated. Fortunately, it is seldom necessary to work with the complete field expressions because one is often interested in the field far from the dipole which is governed predominantly by the terms involving $1/r$. Thus from 5 and 4, we find that for a Hertzian dipole of length $dl$ oriented along the $z$-axis and carrying current

$$I = I_0 \cos \omega t$$  \hspace{1cm} (11)
the electric and magnetic fields at values of $r$ far from the dipole are given by

\[
E = -\frac{\beta^2 I_0 dl \sin \theta}{4\pi \varepsilon \omega r} \sin(\omega t - \beta r) i_\theta \\
= -\frac{\eta \beta I_0 dl \sin \theta}{4\pi r} \sin(\omega t - \beta r) i_\theta
\]

(12a)

\[
H = -\frac{\beta I_0 dl \sin \theta}{4\pi r} \sin(\omega t - \beta r) i_\phi
\]

(12b)

These fields are known as the radiation fields, since they are the components of the total fields that contribute to the time-average radiated power away from the dipole.
Before we discuss the nature of these fields, let us find out quantitatively what we mean by far from the dipole.

To do this, we look at the expression for the complete magnetic field given by 4 and note that the ratio of the amplitudes of the $1/r^2$ and $1/r$ terms is equal to $1/\beta r$.

Hence for $\beta r \gg 1$, the $1/r^2$ term is negligible compared to the $1/r$ term as already pointed out.

This means that for $r \gg 1/\beta$, or $r \gg \lambda/2\pi$, that is, even at a distance of a few wavelengths from the dipole, the fields are predominantly radiation fields.
Returning now to the expressions for the radiation fields given by 12a and 12b, we note that at any given point, (1) the electric field \( E_\theta \), the magnetic field \( H_\phi \), and the direction of propagation \( r \) are mutually perpendicular and (2) the ratio of \( E_\theta \) to \( H_\phi \) is equal to \( \eta \), which are characteristic of uniform plane waves.

The phase of the field, however, is uniform over the surfaces \( r = \text{constant} \), that is, spherical surfaces centered at the dipole, whereas the amplitude of the field is uniform over surfaces \( (\sin \theta)/r = \text{constant} \).
Hence the fields are only locally uniform plane waves, that is, over any small area normal to the $r$-direction at a given point.

The Poynting vector due to the radiation frequency fields is given by

\[
P = E \times H = E_\theta i_\theta \times H_\phi i_\phi = E_\theta H_\phi i_r
\]

\[
= \frac{\eta \beta^2 I_0^2 (dl)^2 \sin^2 \theta}{16\pi^2 r^2} \sin^2(\omega t - \beta r) i_r
\]

(13)
By evaluating the surface integral of the Poynting vector over any surface enclosing the dipole, we can find the power flow out of that surface, that is, the power **radiated** by the dipole.

For convenience in evaluating the surface integral, we choose the spherical surface of radius $r$ and centered at the dipole.

Thus noting that the differential surface area on the spherical surface is $(r d\theta)(r \sin \theta d\phi) i_r$ or $r^2 \sin \theta d\theta d\phi i_r$, we obtain the instantaneous power radiated to be
\[ P_{rad} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} P \cdot r^2 \sin \theta d\theta d\phi I_r \]

\[ P_{rad} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \eta \beta^2 I_0^2 (dl)^2 \frac{\sin^3 \theta}{16\pi^2} \sin^2(\omega t - \beta r) d\theta d\phi \]

\[ = \frac{\eta \beta^2 I_0^2 (dl)^2}{8\pi} \sin^2(\omega t - \beta r) \int_{\theta=0}^{\pi} \sin^3 \theta \ d\theta \]

\[ = \frac{\eta \beta^2 I_0^2 (dl)^2}{6\pi} \sin^2(\omega t - \beta r) \]

\[ = \frac{2\pi \eta I_0^2}{3} \left( \frac{dl}{\lambda} \right)^2 \sin^2(\omega t - \beta r) \]

(14)
The time-average power radiated by the dipole, that is, the average of $P_{rad}$ over one period of the current variation, is

\[
\langle P_{rad} \rangle = \frac{2\pi \eta I_0^2}{3} \left( \frac{dl}{\lambda} \right)^2 \langle \sin^2(\omega t - \beta r) \rangle
\]

\[
= \frac{\pi \eta I_0^2}{3} \left( \frac{dl}{\lambda} \right)^2
\]

\[
= \frac{1}{2} I_0^2 \left[ \frac{2\pi \eta}{3} \left( \frac{dl}{\lambda} \right)^2 \right]
\]
Radiation Fields Due to Hertzian Dipole

Radiation Resistance

We now define a quantity known as the radiation resistance of the antenna, denoted by the symbol $R_{rad}$, as the value of a fictitious resistor that dissipates the same amount of time average power as that radiated by the antenna when a current of the same peak amplitude as that in the antenna is passed through it.

Recalling that the average power dissipated in a resistor $R$ when a current $I_0 \cos \omega t$ is passed through it is $1/2I_0^2R$, we note from 15 that the radiation resistance of the Hertzian dipole is
As a numerical example, for \((dl/\lambda)\) equal to 0.01, 
\[ R_{rad} = 80\pi^2(0.01)^2 = 0.08\Omega. \]
Thus for a current of peak amplitude 1A the 
time-average radiated power is equal to 0.04W.
This indicates that a Hertzian dipole of length 
0.01\(\lambda\) is not a very effective radiator.
We note from ?? that the radiation resistance 
and hence the radiated power are proportional 
to the square of the electrical length, that is, the 
physical length expressed in terms of 
wavelength, of the dipole.
The result given by ?? is, however, valid only for 
small values of \(dl/\lambda\) since if \(dl/\lambda\) is not small, the 
amplitude of the current along the antenna can 
no longer be uniform and its variation must be
It is customary to depict the radiation characteristic by means of a radiation pattern, which can be obtained by shrinking the radius of the spherical surface to zero with the Poynting vectors attached to it and then joining the tips of the Poynting vectors.

Thus the distance from the dipole point to a point on the radiation pattern is proportional to the power density in the direction of that point.

Similarly, the radiation pattern for the fields can be drawn, based upon the $\sin \theta$ dependence of the fields.
In view of the independence of the fields from $\phi$, the patterns are valid for any plane containing the axis of the dipole.

In fact, the three-dimensional radiation patterns can be imagined to be the figures obtained by revolving these patterns about the dipole axis.

For a general case, the radiation may also depend on $\phi$, and hence it will be necessary to draw a radiation pattern for the $\theta = \pi/2$ plane.

Here, this pattern is merely a circle centered at the dipole.
Figure 1: Radiation Pattern of a Hertzian Dipole
Figure 2: Radiation Pattern of a Hertzian Dipole in dB
Figure 3: Cross Sectional View of Radiation Pattern of a Hertzian Dipole
Figure 4: Cross Sectional View of Radiation Pattern of a Hertzian Dipole
Reference

Nannapaneni Narayana Rao.

*Elements of Engineering Electromagnetics.*