A General $z$-Transform Formula for Sampled-Data Systems

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Abstract—In this paper a generalized formulation for the convolution $z$ transform is presented. The advent of this formulation could cover most of the sampled-data cases discussed in the literature. This includes nonsynchronized, cyclic rate, multirate, and nonsynchronized multirate sampling. Furthermore, the use of this formula lies in its applications to certain forms of nonlinear discrete analysis, as well as to time-varying cases. Each of the applications is singled out, and the proper references are cited.

I. INTRODUCTION

The purpose of this paper is to introduce a general, or unified, $z$-transform formula that can be applied to many sampling schemes in feedback systems. It is shown that this formula covers most of the encountered forms of sampled-data systems, as well as having many other applications. This unified theorem is based on a slight generalization of the convolution $z$-transform theorem, which has been recently studied by this author as well as others.\(^1\)

In the past, the convolution formula has been derived for $\Delta = \Delta_0 = 0$ and also when $\Delta \neq 0$. The extension for $\Delta \neq 0$ is quite important due to the fact that many more sampling schemes can be tackled. The main contribution of this paper is based on this extension and lies also in pointing out the various applications.

Convolution Formulas

The following formulation is divided into two parts. Part A covers the $z$-transform case, and Part B covers the modified $z$ transform. Though Part A can be obtained from Part B by a limiting process, its representation in the present form is advantageous in applications.

Part A

\[ ZL[f(t-\Delta_0 T)h(t-\Delta_0 T)] = L[f(t)h(t-\Delta_0 T) + \delta f(t-\Delta_0 T) + \delta h(t-\Delta_0 T)], \quad 0 \leq \delta < 1 \]

\[ = e^{-\delta \nu T} \int_0^\infty \nu \mathcal{F}(\nu, \Delta_0 T)S(\nu, t) df, \quad 0 \leq \Delta_0 \leq \delta, \]

\[ = e^{-\delta \nu T} \int_0^\infty \nu \mathcal{F}(\nu, 1 + \delta - \Delta_0)S(\nu, t) df, \quad \delta \leq \Delta_0 \leq 1 + \delta, \]

\[ = e^{-\delta \nu T} \int_0^\infty \nu \mathcal{F}(\nu, 1 + \Delta_0)S(\nu, t) df, \quad 0 \leq \Delta_0 \leq \delta, \]

where in (1) \((5)\)

\[ \mathcal{F}(s, m) = Z\mathcal{F}(s, m)|_{m=1}, \quad \mathcal{F}(s, m) = Z\mathcal{F}(s, m)|_{m=m}, \]

and the modified $z$ transforms on the right-hand sides are the usual modified $z$ transforms, i.e., $\mathcal{F}(s, m)$ is the $z$ transform of $\mathcal{F}(s, m)$.\(^2\)

Part B

\[ ZL[f(t-\Delta_0 T)h(t-\Delta_0 T)] = L[f(t-\Delta_0 T)h(t-\Delta_0 T)], \quad m \leq \Delta_0 \leq 1 \]

\[ = e^{-\delta \nu T} \int_0^\infty \nu \mathcal{F}(\nu, m + \Delta_0)S(\nu, t) df, \quad 0 \leq \Delta_0 \leq 1 + \delta, \]

\[ = e^{-\delta \nu T} \int_0^\infty \nu \mathcal{F}(\nu, 1 + \Delta_0)S(\nu, t) df, \quad \delta \leq \Delta_0 \leq 1 + \delta, \]

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II. APPLICATIONS

1) Delayed Samplers Operations (Nonsynchronized Sampling)

This case occurs when two samplers in the system are not synchronized, or when the sampler operation is not synchronized with the time application of the signal. It is obtained from (1) by letting $\Delta_0 = \Delta_0 = 0$ and $h(t) = \delta(t) = \text{unit step}$. Therefore,

\[ ZL[f(t)] = e^{-\delta \nu T} \int_0^\infty \nu \mathcal{F}(\nu, \Delta_0 T) df, \quad 0 \leq \Delta_0 \leq 1 + \delta, \]

\[ = e^{-\delta \nu T} \int_0^\infty \nu \mathcal{F}(\nu, \Delta_0 T) df, \quad \Delta_0 \leq 1 + \delta, \]

The mathematical operation is presented symbolically in Fig. 1.

2) Integer Rate Identity (Skip Sampling)\(^3\)

In this case, as shown in Fig. 2, one can obtain the $z$ transform with respect to the period $\nu T$ from the $z$ transform of period $T$. From (1), letting $\Delta_0 = \Delta_0 = 0 = \text{constant}$, and $h(t) = \delta(t) = \text{unit step}$,

\[ \mathcal{F}(\nu, m) = \frac{1}{2\nu T} \int_0^{2\nu T} \mathcal{F}(\nu, t) dt = \frac{1}{2\nu T} \int_0^{2\nu T} \mathcal{F}(\nu, t) dt = \frac{1}{2\nu T} \int_0^{2\nu T} \mathcal{F}(\nu, t) dt. \]

Evaluating the above equation by enclosing the poles of $1/(1-\nu T)^{-m}$, one obtains,

\[ \mathcal{F}(\nu, m) = \frac{1}{2\nu T} \int_0^{2\nu T} \mathcal{F}(\nu, t) dt = \mathcal{F}(\nu, m) = Z\mathcal{F}(\nu, t) = e^{-\delta \nu T}. \]

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\(1\) $\Delta_0, \Delta_0$ are delays in the time functions $f(t)$ and $h(t)$ [see (1)].

\(2\) A delay is a delay in the string of unit impulses, with respect to the time origin [see (1)].

\(3\) Where $h(t-\Delta_0 T)$ is more briefly denoted by $f(t, m)$.
3) Submultiple of the Sampling Period

As shown in Fig. 3, one can obtain the z transform of \( f(t) \) for the period \( T' = T/n \) from the z transform of period \( T \) as follows.

From (1), letting \( \Delta t = \Delta t_1 = \delta = 0 \), \( h(t) = f(t) \), \( f(t) = u(t) \),

\[
F(z) = \frac{1}{2\pi j} \int_T^t \frac{f(u)\,du}{p_u - 1} F(z/p_u)\,dp_u.
\]

However,

\[
F(z/p_u) = F((z/p_u)^n),
\]

where

\[
z = z^n_u = e^{(T/n)\alpha}. \tag{19}
\]

Hence

\[
F(z) = F(z^n_u). \tag{20}
\]

Also, the above reduces to \( F(z) \) if \( z^n_u = z \).

4) Rational Rate Identity (Multirate Sampled Systems)\[11\]

This is a case of two samplers with periods \( n_1T \) and \( n_2T \), where \( n_1/m \) is a rational number (Fig. 4).

In this case, it is often required to find the z transform of period \( n_1T \) in terms of z transform of period \( T \). This can be readily obtained by repeated application of the integer rate identity as follows:

\[
C(z_m, \delta) = e^{-z_m/n_1} \sum_{m=0}^{n_1-1} \sum_{l=0}^{n_2-1} F(z_m e^{(T/n_1)\alpha} e^{l(T/n_2)\alpha}). \tag{21}
\]

5) Delayed Multirate Samplers\[11\]

As shown in Fig. 5, one can obtain the z transform of period \( n_1T \) in terms of period \( T \). This can be achieved by using the results of

\[\text{This identity is useful in the study of general multirate sampled-data systems.}\[20\]
8) Frequency Domain Sampler Decomposition

In some applications such as the study of multiloop, multirate sampled-data systems, the multirate identity (21) is often used in the frequency domain. By noting Remark 6, write (21) as follows:

\[
\{C(z)\}^{*z^p} = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{k=0}^{n-1} F^k \left( s + j\frac{2\pi k}{n+1} \right) \left( s + j\frac{2\pi k}{n+1} \right)^p.
\] (25)

9) Modified z Transform of the Integral Time Function

By using (7) of Part B of the theorem and letting \( \delta_1 = \delta_z = 0 \), one can obtain

\[
\mathcal{Zm} \left[ \int_0^t \mathcal{F}(t) dt \right] = \frac{T}{2} \int_0^\infty \mathcal{F}(p, \delta) \mathcal{F}(\delta, p) dp dm + \frac{T}{2} \int_0^\infty \mathcal{F}(p, x) \mathcal{F}(\delta, p) dp dx.
\] (26)

10) To Obtain the Inverse \( \mathcal{F}(z) - F(z) \) Transformation

This transformation is given as follows:

\[
F(z) = \mathcal{F}^{-1}[\mathcal{F}(z)] = \frac{T}{2} \int_0^\infty \mathcal{F}(p, \delta) \mathcal{F}(\delta, p) \frac{1}{s - q} dq.
\] (27)

11) To Obtain the z Transform and Modified z Transform of Special Functions

It is of interest to note that, letting \( \delta_1 = \delta_z = 0 \) and \( h(t) = u(t) \) = unit step, then (1) and (7) yield the \( z \) transform and the modified \( z \) transform, respectively.

12) To Obtain Summation of Infinite and Finite Series

This summation is obtained from (1) by letting \( \delta_1 = \delta_z = 0 \) and \( s = 1 \). Hence

\[
\sum_{n=-\infty}^\infty \delta_n = \frac{1}{2T} \int_0^\infty \mathcal{F}(p) \mathcal{F}(p - 1) dp.
\] (28a)

For the finite summation, let \( h(t) \) be a finite function, i.e.,

\[
h(t) = 1 \quad \text{for } n \in [0, k],
\]

\[
h(t) = 0 \quad \text{otherwise}
\] (28b)

to obtain

\[
\sum_{n=0}^k \delta_n = \frac{1}{2T} \int_0^\infty \mathcal{F}(p) \left( \frac{p^{k+1} - 1}{p - 1} \right) dp.
\] (29)

13) Evaluation of Integrals or Moments of Time Functions

From (7) of Part B, one can determine the integral of time function as follows:

\[
\int_0^\infty \mathcal{F}(t) dt = \frac{T}{2} \int_0^\infty \mathcal{F}(p, \delta) \mathcal{F}(\delta, p) dp dm.
\] (30)

Similarly, one can evaluate moments of the form

\[
\int_0^\infty \mathcal{F}^k(t) dt
\]

and finite integrals of the form

\[
\int_0^t h(t) dt.
\]

The evaluation of the integral of the square of the time function or of the moments is important in the design of sampled-data systems.

14) Application in the Analysis of Finite Pulsed Feedback Systems with Periodically Varying Sampling Rate and Pulse Width

This equation is also referred to as frequency-domain sampled decomposition in the \( \delta \) plane. A useful relationship between time and frequency domain decompositions is obtained.

15) Unmixed Difference-Differential and Sum Equations

The convolution \( \mathcal{F} \)-transform is applicable to the solution of equations of the following form

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{F}(n) + \sum_{k=0}^{\infty} \mathcal{F}(n) = \mathcal{F}(0).
\] (31)

16) In Obtaining Periodic Modes of Oscillation in Nonlinear Sampled-Data Systems

The use of the convolution formula in the limit cycle analysis of certain nonlinear discrete systems is established. It is particular in it is used in the study of PWM (pulse width modulated), quantized, relay, and saturating systems and also systems with hysteresis.

17) In the Solution of Certain Nonlinear Sampled-Data Systems

This method is suitable for obtaining analytic solution of systems described by the following nonlinear difference equation

\[
\sum_{p=0}^{\infty} h(t + p) + f(x(n), x(n + 1), \ldots, x(n + r - 1)) = 0.
\] (32)

18) In the Solution of Certain Sampled-Data Systems with Periodically Varying Parameters

III. Conclusion

In this paper a general convolution theorem is presented in its various forms. The application of this formula is illustrated in various cases which cover a substantial part of sampled-data investigations. The purpose of this paper can be summarized as follows.

1) The introduction of the general formula is quite useful in the teaching of sampled-data or in self study. It is important in the fact that many cases can be treated from this formula and thus shorten the time for discussing the various cases.

2) If the practicing engineer in industry is faced with a new sampling scheme, then the formula might supply either the answer or the way to obtain the solution.

3) For the research worker, the formula may indicate the various cases which have not been covered in the literature and which are worthwhile to investigate.

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References


